

Hyperbolicity of Generic High-Degree Hypersurfaces in Complex Projective Spaces

Dedicated to the Memory of Hans Grauert

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CONTENTS

0. INTRODUCTION	1
1. APPROACH OF VECTOR FIELDS AND LIE DERIVATIVES	13
2. CONSTRUCTION OF SLANTED VECTOR FIELDS FOR JET SPACE	19
3. CONSTRUCTION OF HOLOMORPHIC JET DIFFERENTIALS	48
4. HYPERBOLICITY FROM SLANTED VECTOR FIELDS AND VANISHING ORDER OF LOW-DEGREE POLYNOMIALS	62
5. ESSENTIAL SINGULARITIES, VARYING COEFFICIENTS, AND SECOND MAIN THEOREM	72
References	92

0. INTRODUCTION

In this paper we are going to present the proofs of the following two theorems on the hyperbolicity of generic hypersurfaces of sufficiently high degree and of their complements, together with a number of related results, obtained by the same methods, such as: (i) a Big-Picard-Theorem type statement concerning extendibility, across the puncture, of holomorphic maps from a punctured disk to a generic hypersurface of high degree, (ii) entire holomorphic functions satisfying polynomial equations with slowly varying coefficients, and (iii) Second Main Theorems for jet differentials and slowly moving targets.

Theorem 0.1. *For any integer $n \geq 3$ there exists a positive integer δ_n (which is explicitly expressible as a function of n) with the following property. For any generic hypersurface X in \mathbb{P}_n of degree $\delta \geq \delta_n$ there is no nonconstant holomorphic map from \mathbb{C} to X .*

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Theorem 0.2. *For any integer $n \geq 2$ there exists a positive integer δ_n^* (which is explicitly expressible as a function of n) with the following property. For any generic hypersurface X in \mathbb{P}_n of degree $\delta \geq \delta_n^*$ there is no nonconstant holomorphic map from \mathbb{C} to $\mathbb{P}_n - X$.*

Theorem 0.1 was presented with a sketch of its proof in [Si02] and [Si04]. The methods used, though rather tedious in some of their details, consist essentially just of some skillful manipulations in linear algebra and the chain rule of differentiation. The underlying ideas in these methods can be traced to the techniques which Bloch developed in his 1926 paper [B26]. To explain this link to Bloch's paper [B26], we first very briefly describe Bloch's techniques with explanations about how they foreshadow to a certain extent our techniques in this paper.

0.3. Bloch's Technique of Construction of Jet Differential.

In his 1926 paper [B26] Bloch proved the nonexistence of nonconstant holomorphic maps from \mathbb{C} to a submanifold Y of an abelian variety A , which does not contain a translate of a positive-dimensional abelian subvariety of A , by producing sufficiently independent holomorphic jet differentials ω on Y vanishing on some ample divisor of Y and using the fact that the pullbacks of such jet differentials by holomorphic maps from \mathbb{C} to Y vanish identically.

He produced such holomorphic jet differentials ω on Y , not by applying to Y the theorem of Riemann-Roch (which was not yet readily available at the time of Bloch's paper for the case needed for its application to Y), but explicitly by pulling back to Y constant-coefficient polynomials P (with homogeneous weight) of differentials of the coordinates (including higher-order differentials) of the universal cover \tilde{A} of the abelian variety A . When the constant-coefficient polynomials P of differentials of coordinates of \tilde{A} are pulled back to Y , the condition of Y not containing a translate of a positive-dimensional abelian subvariety of A causes new vanishing of the pullbacks on Y . Moreover, the new vanishing is on some ample divisor of Y when the constant-coefficient polynomials P of differentials of coordinates of \tilde{A} are appropriately chosen. The reason why it is possible to choose P so that its pullback ω to Y vanishes on an ample divisor of Y is that the condition of not containing a translate of a positive-dimensional abelian subvariety of A guarantees that the dimension of the \mathbb{C} -vector space of the pullbacks to Y of all possible such polynomials P is so high that at least one \mathbb{C} -linear combination ω of such pullbacks vanishes on some ample divisor of Y .

Bloch's construction is related to the classical construction of a \mathbb{C} -basis of holomorphic 1-forms for a regular plane curve C defined by an equation $R(x, y) = 0$ of degree $\delta \geq 3$ in the inhomogeneous coordinates x, y of \mathbb{P}_2 , which are constructed by pulling back to C meromorphic 1-forms

$$P(x, y) \frac{dx}{R_y(x, y)} = P(x, y) \frac{-dy}{R_x(x, y)}$$

of “low pole order” on \mathbb{P}_2 , where $R_x(x, y)$ and $R_y(x, y)$ are the first-order partial derivatives of $R(x, y)$ and $P(x, y)$ is a polynomial of degree $\leq \delta - 3$. The adjunction formula for the plane curve C causes new vanishing to cancel the “low pole order” of the meromorphic 1-forms on \mathbb{P}_2 to yield holomorphic 1-forms on C when the meromorphic 1-forms on \mathbb{P}_2 are pulled back to the plane curve C .

In this paper, the construction of holomorphic jet differentials on a generic hypersurface X of sufficiently high degree δ in \mathbb{P}_n combines Bloch's method and the classical construction of holomorphic 1-forms on plane curves of high degree. We take meromorphic jet differentials of low pole orders (of magnitude $\delta^{1-\varepsilon}$ for some appropriate $0 < \varepsilon < 1$) on \mathbb{P}_n and pull them back to X . The high degree δ of X will guarantee (according to Lemma 3.4 concerning the injectivity of the pullback map for certain jet differentials) that the dimension of the \mathbb{C} -vector space of such pullbacks is so high that some \mathbb{C} -linear combination of such pullbacks will be a non identically zero holomorphic jet differential on X vanishing on some ample divisor of X (see Proposition 3.8 below). One key point in this argument is that, because the dimension of \mathbb{P}_n is higher than that of X , there are more degrees of freedom in constructing meromorphic $(n-1)$ -jet differentials of low pole order on \mathbb{P}_n and, if the pullback map to X of such meromorphic $(n-1)$ -jet differentials is injective, there are sufficient independent pullbacks to X to form a non identically zero \mathbb{C} -linear combination which vanishes on an ample divisor of X .

0.4. Key Technique of Slanted Vector Fields.

After so many decades of impasse, the real key which opens the way to the proof of the hyperbolicity of generic hypersurface of sufficiently high degree (in the sense stated in Theorem 0.1) is the introduction in [Si02][Si04] of the technique of slanted vector fields in the subspace $J_{n-1}^{(\text{vert})}(\mathcal{X})$ of vertical $(n-1)$ -jets in the $(n-1)$ -jet space $J_{n-1}(\mathcal{X})$ of the universal hypersurface \mathcal{X} of degree δ in $\mathbb{P}_n \times \mathbb{P}_N$ (where $N = \binom{\delta+n}{n} - 1$).

For a complex manifold Y the space $J_k(Y)$ of k -jets of Y consists of all k -jets of Y (each of which is represented by a parametrized complex

curve germ). The universal hypersurface \mathcal{X} of degree δ in $\mathbb{P}_n \times \mathbb{P}_N$ (with $N = \binom{\delta+n}{n} - 1$) is defined by

$$(0.4.1) \quad \sum_{\nu_0 + \dots + \nu_n = \delta} \alpha_{\nu_0, \dots, \nu_n} z_0^{\nu_0} \cdots z_n^{\nu_n} = 0,$$

where $\alpha = [\alpha_{\nu_0, \dots, \nu_n}]_{\nu_0 + \dots + \nu_n = \delta}$ is the homogeneous coordinate of \mathbb{P}_N and $[z_0, \dots, z_n]$ is the homogeneous coordinate of \mathbb{P}_n . For $\alpha \in \mathbb{P}_N$ let $X^{(\alpha)}$ be the hypersurface of degree δ defined by (0.4.1) when α is fixed as constant. A vertical k -jet of \mathcal{X} is a k -jet in \mathcal{X} representable by some (parametrized) complex curve germ lying completely in some fiber $X^{(\alpha)}$ of \mathcal{X} . We denote by $J_k^{(\text{vert})}(\mathcal{X})$ the space of all vertical k -jets on \mathcal{X} . There is a projection map $\pi_{k, \text{vert}} : J_k^{(\text{vert})}(\mathcal{X}) \rightarrow \mathbb{P}_N$ such that an element P_0 of $J_k^{(\text{vert})}(\mathcal{X})$ is represented by a (parametrized) complex curve germ in $X^{(\alpha)}$ with $\alpha = \pi_{k, \text{vert}}(P_0)$.

A slanted vector field ξ on $J_k^{(\text{vert})}(\mathcal{X})$ means a vector field of $J_k^{(\text{vert})}(\mathcal{X})$ which at a generic point P_0 of $J_k^{(\text{vert})}(\mathcal{X})$ is not tangential to the space $J_k(X^{(\alpha)})$ of k -jets of the fiber $X^{(\alpha)}$ at the point P_0 of $J_k(X^{(\alpha)})$ with $\alpha = \pi_{k, \text{vert}}(P_0)$. When a local k -jet differential form on $X^{(\alpha)}$ defined for α in some open subset U of \mathbb{P}_N is regarded as a local function on $J_k^{(\text{vert})}(\mathcal{X})$ and is differentiated with respect to ξ , the result is a local function on $J_k^{(\text{vert})}(\mathcal{X})$ which is represented by a local k -jet differential on $X^{(\alpha)}$ for $\alpha \in U$. In the case of $k = n - 1$, meromorphic slanted vector fields ξ of low vertical pole-order (of the magnitude $\mathcal{O}_{\mathbb{P}_n}(n^2)$ on the vertical fiber), whose existence is given in Proposition 2.17 below, play the following indispensable role in generating sufficiently independent holomorphic $(n - 1)$ -jet differentials on $X^{(\alpha)}$ vanishing on an ample divisor for a generic α and for δ sufficiently large.

On a regular hypersurface $X^{(\alpha)}$ of high degree δ , there cannot be any nonzero meromorphic vector fields on $X^{(\alpha)}$ of low pole order. However, the universal hypersurface \mathcal{X} of degree δ in $\mathbb{P}_n \times \mathbb{P}_N$ has bidegree $(\delta, 1)$ with respect to the two hyperplane section line bundles $\mathcal{O}_{\mathbb{P}_n}(1)$ and $\mathcal{O}_{\mathbb{P}_N}(1)$. Because of the second component 1 in the bidegree $(\delta, 1)$ of \mathcal{X} , when slanted vector fields are used, it is possible to get meromorphic slanted vector fields of $J_k^{(\text{vert})}(\mathcal{X})$ with low vertical pole order.

For a generic $\hat{\alpha} \in \mathbb{P}_n$ the holomorphic $(n - 1)$ -jet differential $\omega^{(\hat{\alpha})}$ on $X^{(\hat{\alpha})}$ vanishing on an appropriate ample divisor (constructed by pulling back an appropriate meromorphic $(n - 1)$ -jet differential of low pole order on \mathbb{P}_n according to Proposition 3.8 below) can be extended

to a holomorphic family $\omega^{(\alpha)}$ on $X^{(\alpha)}$ for α in some open neighborhood U of $\hat{\alpha}$ in \mathbb{P}_N so that successive application of different finite sets of meromorphic slanted vector fields ξ_1, \dots, ξ_ℓ of low vertical pole order (as constructed in Proposition 2.17 below) would yield sufficiently independent holomorphic jet differentials vanishing on ample divisor on $X^{(\hat{\alpha})}$ so that the application of the Schwarz lemma of the vanishing of pullbacks, to \mathbb{C} by a holomorphic map $\mathbb{C} \rightarrow X^{(\hat{\alpha})}$, of holomorphic jet differentials vanishing on ample divisor of $X^{(\hat{\alpha})}$ would force every holomorphic map from \mathbb{C} to $X^{(\hat{\alpha})}$ to be constant (see Lemma 4.1, Proposition 4.3, and the proof of Theorem 0.1 given in 4.4 below).

We would like to remark that the Schwarz lemma of the vanishing of pullbacks, to \mathbb{C} by a holomorphic map $\mathbb{C} \rightarrow X^{(\hat{\alpha})}$, of holomorphic jet differentials vanishing on ample divisor of $X^{(\hat{\alpha})}$ also has its origin in Bloch's 1926 paper [B26], though its formulation and its proof there are in a form very different from our current way of mathematical presentation. Bloch's proof of applying Nevanlinna's logarithmic derivative lemma (p.51 of [Ne25]) to local coordinates which are the logarithms of global meromorphic functions still is the best proof of the Schwarz lemma. It is recast in the current language of mathematical presentation on pp.1162-1164 of [SY97].

0.5. *Slanted Vector Fields and Bloch's Technique of Maps by Translation.*

The technique of slanted vector fields in a way also finds some remote ancestry in the 1926 paper of Bloch [B26], though the connection is not so transparent. We point this out here in order to dispel the wrong perception that the technique of slanted vector fields is applicable to generic hypersurfaces of high degree because of the variation of complex structure of hypersurfaces as the coefficients of their defining functions vary.

In his proof of the hyperbolicity of a submanifold Y in an abelian variety A which contains no translate of positive-dimensional abelian subvariety of A , Bloch used the vector fields from translations in A to do the differentiation of jet differentials. That is the reason why when such differentiations cannot yield enough independent holomorphic jet differentials to give hyperbolicity of Y , Y must contain a translate of some positive-dimensional abelian subvariety of A . This link of the use of slanted vector fields to Bloch's technique of using vector fields of maps by translation is obscured by the fact that in Bloch's technique the result of differentiation with respect to vector fields of maps by translation is just the same as the use of a different constant-coefficient

polynomial in differentials of the coordinates of the universal cover \tilde{A} of A .

Let us return to our situation at hand of using slanted vector fields ξ on $J_k(X^{(\alpha)})$ of low vertical pole order for $k = n - 1$. Though the complex structure of the hypersurface $X^{(\alpha)}$ changes as α varies in \mathbb{P}_N , the slanted vector fields ξ in general do not respect the fibers in the sense that for two distinct points P_0 and P'_0 on $J_k(X^{(\alpha)})$ the two projections $\pi_{k,\text{vert}}(\xi_{P_0})$ and $\pi_{k,\text{vert}}(\xi_{P'_0})$ are in general different vectors in the tangent space of \mathbb{P}_N at α .

For our situation at hand, the geometric picture is not the pulling back of k -jet differential from a neighboring fiber by the slanted vector field ξ , even in the infinitesimal setting. What is relevant is the existence of slanted vector fields pointing in sufficiently many different directions on $J_k^{(\text{vert})}(\mathcal{X})$ at the prescribed point in question. The realization of the irrelevancy of the variation of the complex structure $X^{(\alpha)}$ as α varies in \mathbb{P}_N , as well as the interpretation of Bloch's technique of differentiation with respect to vector fields of maps by translation, points to the promise of the applicability of our method even to the case of some rigid complex manifolds Z inside some \mathbb{P}_m as a submanifold of possibly high codimension. In certain cases, though Z may be rigid as a compact complex manifold, yet there is a possibility that appropriate meromorphic vector fields on \mathbb{P}_n applied to pullbacks to Z of low pole-order meromorphic jet differentials on \mathbb{P}_n may yield sufficiently independent holomorphic jet differentials on Z vanishing on an ample divisor.

0.6. Necessity of Use of Vertical Jet Space.

The reason why the more complicated space $J_{n-1}^{(\text{vert})}(\mathcal{X})$ of vertical $(n-1)$ -jets of \mathcal{X} has to be used instead of the simpler $(n-1)$ -jet space $J_{n-1}(\mathcal{X})$ of \mathcal{X} is that, while it is possible to extend a holomorphic $(n-1)$ -jet differential $\omega^{(\hat{\alpha})}$ on a hypersurface $X^{(\hat{\alpha})}$ for a generic $\hat{\alpha} \in \mathbb{P}_N$ to a holomorphic family of $\omega^{(\alpha)}$ on $X^{(\alpha)}$ for α in an open neighborhood U of $\hat{\alpha}$ in \mathbb{P}_N , it is in general impossible to find a holomorphic $(n-1)$ -jet differential on the part of $J_{n-1}(\mathcal{X})$ above some open neighborhood U of $\hat{\alpha}$ in \mathbb{P}_N whose pullback to $X^{(\hat{\alpha})}$ is equal to $\omega^{(\hat{\alpha})}$.

Difficulty of the latter kind of extension can be illustrated easily in the case of a holomorphic family of plane curves C_a given by $R(x, y, a) = 0$ with a in the open unit disk of \mathbb{C} as a holomorphic parameter. A

holomorphic 1-form on a single plane curve C_a can be constructed as

$$\frac{dx}{R_y(x, y, a)} = -\frac{dy}{R_x(x, y, a)}$$

from the vanishing of the differential $dR = R_x dx + R_y dy$ on C_a when a is considered as a constant, but in the total space $\bigcup_{a \in \Delta} C_a$ of the family of plane curves it is not easy to carry out a similar construction, because when a is regarded as a variable, the differential dR becomes $R_x dx + R_y dy + R_a da$ and the same method cannot be applied.

Furthermore, it is for this kind of difficulty of constructing $(n-1)$ -jet differentials on the universal hypersurface \mathcal{X} that the additional condition (5.9.1) _{j} for $1 \leq j \leq n-1$ is introduced into Theorem 5.11 on entire function solutions of polynomial equations with slowly varying coefficients, so that families of vertical $(n-1)$ -jet differentials on the fibers can be used instead.

0.7. Algebraic Geometric Counterpart of Slanted Vector Fields.

In his 1986 paper [Cl86] Clemens introduced a technique (later generalized by Ein [Ei88], and Voisin [Voi96]) to prove the nonexistence of rational and elliptic curves in generic hypersurfaces of high degree by showing that the normal bundle of one such curve in the family of such curves is globally generated by sections with vertical pole order 1. His technique can be considered the counterpart of our method of slanted vector fields and as a matter of fact serves as motivation for our method.

On its face value Clemens's technique of using normal bundle to estimate the genus of a curve is algebraic in nature and cannot possibly have anything to do with the problem of hyperbolicity of transcendental in nature. Its relevancy was realized for the first time in [Si02] and [Si04] partly because of our seemingly completely unrelated earlier work on the deformational invariance of the plurigenera [Si98] [Si00].

Like jet differentials, pluricanonical sections can be naturally pulled back by a map and, as a result, their Lie differentiation can be naturally defined without specifying any special connection. In a hitherto unsuccessful attempt to study deformational invariance of sections of other bundles associated with the tangent bundle (besides pluricanonical sections), we investigated the obstruction of moving jet differentials out of a fiber in a family of compact complex manifolds and considered their Lie derivatives with respect to slanted vector fields. Such investigations, though unsuccessful so far as its original goal is concerned, serendipitously led to the use of slanted vector fields in the study of

hyperbolicity problems and to the realization that Clemens's technique is relevant to, and can serve as motivation for, the differentiation of jet differentials by slanted vector fields to produce new ones.

0.8. *Linear Algebra versus Theorem of Riemann-Roch.*

As already pointed out in the paragraph straddling p.445 and p.446 in [Si02], a non identically zero holomorphic $(n - 1)$ -jet differential on $X^{(\alpha)}$ vanishing on an ample divisor can be constructed from the theorem of Riemann-Roch by using the sufficient positivity of the canonical line bundle of $X^{(\alpha)}$ and the lower bound of the negativity of jet differential bundles of $X^{(\alpha)}$. Such a jet differential can also be directly obtained by using the linear algebra method of solving a system of linear equations with more unknowns than independent linear equations, which is the method used here in Proposition 3.8 below, as sketched on p.446 of [Si02]. This direct method of construction by linear algebra has the important advantage of better control over the form of the resulting jet differential so that the application of slanted vector fields can produce sufficiently independent jet differentials vanishing on an ample divisor of $X^{(\alpha)}$ for a generic point α of \mathbb{P}_N (see Lemma 4.1, Proposition 4.3, and the proof of Theorem 0.1 given in 4.4 below).

Of course, the use of the theorem of Riemann-Roch also uses the linear algebra technique of counting the dimension of sections modules and the dimension of obstructional higher cohomology groups, but the process of going through a labyrinth of exact sequences so obscures the eventual form of the resulting jet differential that not enough control can be retained to get beyond the weaker conclusion that holomorphic maps from \mathbb{C} to $X^{(\alpha)}$ is contained in some proper subvariety of $X^{(\alpha)}$.

Recently Diverio, Merker, and Rousseau in [DMR10] used the theorem of Riemann-Roch to construct a holomorphic jet differential on $X^{(\alpha)}$ vanishing on an ample divisor and then used Merker's work [Me09] involving our method of slanted vector fields to arrive at the conclusion that holomorphic maps from \mathbb{C} to $X^{(\alpha)}$ is contained in some proper subvariety of $X^{(\alpha)}$.

0.9. *Simplified Treatment in Going from Non Zariski Density of Entire Curves to Hyperbolicity.*

In the original sketch of the proof of Theorem 0.1 in [Si02], for the last step discussed on p.447 of [Si02] of going from the non Zariski density of entire curves to hyperbolicity, the method of construction of jet differentials is applied to a hypersurface \hat{X} in $\mathbb{P}_{\hat{n}}$ constructed from the hypersurface $X^{(\alpha)}$ in \mathbb{P}_n with a larger \hat{n} so that more jet differentials

on $X^{(\alpha)}$ can be obtained from jet differentials on \hat{X} . The idea is that the zero-set of the jet differentials constructed on $X^{(\alpha)}$ from linear algebra would be defined by the vanishing of polynomials of low degree in low-order partial derivatives of the polynomial $f^{(\alpha)}$ defining $X^{(\alpha)}$ and, in order to take care of such zero-set, the low-order partial derivatives of f are introduced as additional new variables. The genericity condition of f enters in a certain form of independence of the low-order partial derivatives of f .

In this paper we use a simplified treatment of this step and directly use the genericity condition of f as a certain form of independence of the low-order partial derivatives of f . The key point of the argument, given in Lemma 4.1 and Proposition 4.3, is that the pullback, to a generic $X^{(\alpha)}$, of a meromorphic $(n-1)$ -jet differential on \mathbb{P}_n defined by a low-degree polynomials of the inhomogeneous coordinates of \mathbb{P}_n and their differentials have only low vanishing order at every point of $X^{(\alpha)}$.

0.10. *Techniques Parallel to Those in Gelfond-Schneider-Lang-Bombieri Theory.*

Paul Vojta presented in [Voj87] a formal parallelism between the results in diophantine approximation and those in value distribution theory. Along this line, the techniques presented here for the proof of the hyperbolicity of generic hypersurfaces of sufficiently high degree are, in certain ways, quite parallel to the techniques used for the theory of Gelfond-Schneider-Lang-Bombieri ([Sc34], [Ge34], [La62], [La65], [La66], [Bo70], and [BL70]).

- (i) The construction of holomorphic jet differentials in Proposition 3.8 by solving a system of linear equations with more unknowns than equations is parallel to the use of Siegel's lemma in the theory of Gelfond-Schneider-Lang-Bombieri to construct a polynomial with estimates on its degree and the heights of its coefficients.
- (ii) The requirement that the constructed jet differential vanishing on an ample divisor of high degree in Proposition 3.8 is parallel to the requirement of the vanishing of the constructed polynomial in the theory of Gelfond-Schneider-Lang-Bombieri to high order at certain points.
- (iii) Lemma 3.4 concerning the injectivity of the pullback map for certain jet differentials is parallel to the constructed polynomial in the theory of Gelfond-Schneider-Lang-Bombieri being not identically zero due to the assumption of the degree of transcendence of the given functions.

(iv) The use of Nevanlinna's logarithmic derivative lemma and the use of logarithms of global meromorphic functions as local coordinates in the Schwarz lemma to estimate the contribution from the differentials to be of lower order is parallel to the use of the differential equations in the theory of Gelfond-Schneider-Lang-Bombieri.

Such a parallelism between the techniques used in this paper and those in theory of Gelfond-Schneider-Lang-Bombieri lends support to the preferability of the approach used in this paper for the hyperbolicity problem of hypersurfaces.

0.11. Notations and Terminology.

For $r > 0$ we use Δ_r to denote the open unit disk in \mathbb{C} of radius r centered at the origin. When $r = 1$, we simply use Δ to denote Δ_1 when there is no confusion.

For a real number λ denote by $\lfloor \lambda \rfloor$ the round-down of λ which means the largest integer $\leq \lambda$ and denote by $\lceil \lambda \rceil$ be the round-up of λ which means the smallest integer $\geq \lambda$.

We use $[z_0, \dots, z_n]$ to denote the homogeneous coordinates of \mathbb{P}_n and we use (x_1, \dots, x_n) to denote the inhomogeneous coordinates of \mathbb{P}_n with $x_j = \frac{z_j}{z_0}$ for $1 \leq j \leq n$. Sometimes we also go to the inhomogeneous coordinates by fixing $z_0 \equiv 1$ in the homogeneous coordinates when notationally it is more advantageous to do so.

Denote by \mathbb{N} the set of all positive integers. For an $(n+1)$ -tuple $\nu \in (\mathbb{N} \cup \{0\})^{n+1}$ of nonnegative integers, we write $\nu = (\nu_0, \nu_1, \dots, \nu_n)$ and $|\nu| = \nu_0 + \nu_1 + \dots + \nu_n$ and let

$$z^\nu = z_0^{\nu_0} z_1^{\nu_1} \dots z_n^{\nu_n}.$$

For $0 \leq p \leq n$, let e_p denote the unit vector in \mathbb{C}^{n+1} such that all components are zero except that the component in the p -th place is 1. We use $\delta_{\nu, \mu}$ to denote the Kronecker delta for the indices $\nu, \mu \in (\mathbb{N} \cup \{0\})^{n+1}$, which assumes the value 1 for $\nu = \mu$ and assumes the value 0 when $\nu \neq \mu$.

If from the context there is no risk of confusion, we use N to denote $\binom{\delta+n}{n} - 1$ so that \mathbb{P}_N is the moduli space for all hypersurface of degree δ , without further explicit mention. The homogeneous coordinates of \mathbb{P}_N will be denoted by $\alpha = [\alpha_{\nu_0, \dots, \nu_n}]_{\nu_0 + \dots + \nu_n = \delta}$. The hypersurface defined by

$$f^{(\alpha)} = \sum_{\nu_0 + \nu_1 + \dots + \nu_n = \delta} \alpha_{\nu_0, \dots, \nu_n} z_0^{\nu_0} z_1^{\nu_1} \dots z_n^{\nu_n}$$

is denoted by $X^{(\alpha)}$. For notational simplicity, sometimes the superscript (α) in $f^{(\alpha)}$ and $X^{(\alpha)}$ is dropped when there is no risk of confusion. Also sometimes we simply use $f^{(\alpha)}(x_1, \dots, x_n)$ or $f(x_1, \dots, x_n)$ to mean $\frac{1}{z_0^\alpha} f^{(\alpha)}(z_0, \dots, z_n)$ when the context makes it clear what is being meant. This notational simplification by dropping superscript (α) applies also to other symbols such as replacing $Q^{(\alpha)}$ by Q (respectively $\omega^{(\alpha)}$ by ω) when there is no risk of confusion or replacing Q by $Q^{(\alpha)}$ (respectively ω by $\omega^{(\alpha)}$) when there is a need to keep track of the dependence on the parameter $\alpha \in \mathbb{P}_N$.

When we present the main ideas of an argument and refer to high vanishing order without explicitly giving a precise number, we mean a quantity of the order of δ . In such a situation, when we refer to low pole order without explicitly giving a precise number, we mean a quantity of the order of $\delta^{1-\varepsilon}$ for some appropriate $0 < \varepsilon < 1$.

The notation at the end of the inequality

$$A(r) \leq B(r) \quad \parallel$$

means that there exist $r_0 > 0$ and a subset E of $\mathbb{R} \cap \{r > r_0\}$ with finite Lebesgue measure such that the inequality holds for $r > r_0$ and not in E . This is the condition needed for the logarithmic derivative lemma of Nevanlinna as given at the bottom of p.51 of [Ne25].

For a meromorphic function F on \mathbb{C} and $c \in \mathbb{C} \cup \{\infty\}$ with $F(0) \neq c$ the *counting function* is

$$N(r, F, c) = \int_{\rho=0}^r n(\rho, F, c) \frac{d\rho}{\rho},$$

where $n(\rho, F, c)$ is the number of roots of $F = c$ in Δ_ρ with multiplicities counted. The *characteristic function* is

$$T(r, F) = N(r, F, \infty) + \frac{1}{4\pi} \int_{\theta=0}^{2\pi} \log^+ |F(re^{i\theta})| d\theta$$

under the assumption that 0 is not a pole of F , where \log^+ means the maximum of \log and 0.

For a complex manifold Y with a $(1, 1)$ -form η and for a holomorphic map $\varphi : \mathbb{C} \rightarrow Y$, the characteristic function of φ with respect to η is

$$T(r, \varphi, \eta) = \int_{\rho=0}^r \left(\int_{\Delta_\rho} \varphi^* \eta \right) \frac{d\rho}{\rho}.$$

0.12. *Twice Integration of Laplacian in Nevanlinna Theory.*

The technique of twice integrating the Laplacian of a function introduced by Nevanlinna for his theory of value distribution will be used a number of times in this paper. We put it down here for reference later. For any smooth function $g(\zeta)$, from the divergence theorem

$$\int_{|\zeta| < r} \Delta g = \int_{\theta=0}^{2\pi} \left(\frac{\partial}{\partial r} g(re^{i\theta}) \right) r d\theta$$

and $\Delta = 4\partial_\zeta \partial_{\bar\zeta}$ it follows that

$$\begin{aligned} 4 \int_{\rho=r_1}^r \left(\int_{|\zeta| < \rho} \partial_\zeta \partial_{\bar\zeta} g \right) \frac{d\rho}{\rho} &= \int_{\rho=r_1}^r \left(\int_{|z| < \rho} \Delta g \right) \frac{d\rho}{\rho} \\ &= \int_{\theta=0}^{2\pi} g(re^{i\theta}) d\theta - \int_{\theta=0}^{2\pi} g(r_1 e^{i\theta}) d\theta. \end{aligned}$$

0.13. Function Associated to Pullback of Jet Differential to Part of Complex Line.

Let ω be a holomorphic k -jet differential on a complex manifold Y of complex dimension n and φ be a holomorphic map from an open subset U of \mathbb{C} (with coordinate ζ) to Y . The map φ induces a map $J_{k,\varphi}$ from the space $J_k(U)$ of k -jets on U to the space $J_k(Y)$ of k -jets on Y , which sends a k -jet η on U at ζ_0 represented by a parametrized complex curve germ $\gamma : \Delta \rightarrow U$ with $\gamma(0) = \zeta_0$ to the k -jet represented by the parametrized complex curve germ $\varphi \circ \gamma : \Delta \rightarrow Y$ at $\varphi(\zeta_0)$. The pullback $\varphi^* \omega$ of ω by φ means the holomorphic k -jet differential on U whose value at a k -jet η of U at ζ_0 is the value of ω at the k -jet $J_{k,\varphi}(\eta)$ of $J_k(Y)$ at $\varphi(\zeta_0)$.

The crucial tool in the study of the hyperbolicity problem is the result, usually referred to as the Schwarz lemma, of the vanishing of the pullback of a holomorphic jet differential on a compact complex manifold vanishing on an ample divisor by a holomorphic map from \mathbb{C} . Its proof, by Bloch's technique of using the logarithmic derivative lemma of Nevanlinna (p.51 of [Ne25]) with logarithms of global meromorphic functions as local coordinates, first shows the vanishing of a function associated to the pullback of the jet differential and then obtains the vanishing of the pullback of the jet differential by composing the map from \mathbb{C} with appropriate holomorphic maps $\mathbb{C} \rightarrow \mathbb{C}$.

In the later part of this article, when the analogues of the Big Picard Theorem are introduced for generic hypersurfaces X of high degree to extend holomorphic maps from $\mathbb{C} - \overline{\Delta_{r_0}} \rightarrow X$ to holomorphic maps from $\mathbb{C} \cup \{\infty\} - \overline{\Delta_{r_0}} \rightarrow X$, appropriate holomorphic maps $\mathbb{C} - \overline{\Delta_{r_0}}$ to itself are unavailable for proof the full Schwarz lemma. Instead

only the vanishing of the function associated to the pullback of the jet differential can be obtained. We now introduce a notation for this function. The function on U , denoted by $\text{eval}_{id_{\mathbb{C}}}(\varphi^*\omega)$, at the point ζ_0 is the value of the k -jet $\varphi^*\omega$ evaluated at the k -jet of U at ζ_0 represented by the parametrized curve defined by the identity map of \mathbb{C} . In other words, the value of $\text{eval}_{id_{\mathbb{C}}}(\varphi^*\omega)$ at $\zeta_0 \in U$ is the value of ω at the k -jet on Y represented by the parametrized complex curve germ $\varphi : U \rightarrow Y$ at $\varphi(\zeta_0)$.

When for some local coordinates y_1, \dots, y_n of Y the k -jet differential ω is written as

$$\sum_{\boldsymbol{\nu}} G_{\boldsymbol{\nu}}(y_1, \dots, y_n) \prod_{1 \leq j \leq n, 1 \leq \ell \leq k} (d^\ell y_j)^{\nu_{\ell,j}}$$

where $\boldsymbol{\nu} = (\nu_{\ell,j})_{1 \leq j \leq n, 1 \leq \ell \leq k}$. The function $\text{eval}_{id_{\mathbb{C}}}(\varphi^*\omega)$ at $\zeta \in U$ is given by

$$\sum_{\boldsymbol{\nu}} G_{\boldsymbol{\nu}}(\varphi_1(\zeta), \dots, \varphi_n(\zeta)) \prod_{1 \leq j \leq n, 1 \leq \ell \leq k} \left(\frac{d^\ell}{d\zeta^\ell} \varphi_j(\zeta) \right)^{\nu_{\ell,j}},$$

where φ is represented by $(\varphi_1, \dots, \varphi_n)$ with respect to the local coordinates y_1, \dots, y_n of Y , so that if y_j is locally equal to $\log F_j$ for some global meromorphic function F_j on Y (for $1 \leq j \leq n$), the logarithmic derivative lemma of Nevanlinna (p.51 of [Ne25]) can be applied to $\frac{d^\ell}{d\zeta^\ell} \varphi_j(\zeta) = \frac{d^\ell}{d\zeta^\ell} \log F_j(\varphi(\zeta))$.

1. APPROACH OF VECTOR FIELDS AND LIE DERIVATIVES

1.1. Moduli Space of Hypersurfaces.

The moduli space of all hypersurfaces of degree δ in \mathbb{P}_n is the same as the complex projective space \mathbb{P}_N of complex dimension $N = \binom{\delta+n}{n} - 1$. The defining equation for the universal hypersurface \mathcal{X} in $\mathbb{P}_n \times \mathbb{P}_N$ is

$$f = \sum_{\substack{\boldsymbol{\nu} \in (\mathbb{N} \cup \{0\})^{n+1} \\ |\boldsymbol{\nu}| = \delta}} \alpha_{\boldsymbol{\nu}} z^{\boldsymbol{\nu}}.$$

The number of indices $\boldsymbol{\nu} \in (\mathbb{N} \cup \{0\})^{n+1}$ with $|\boldsymbol{\nu}| = \delta$ is $\binom{\delta+n}{n} = N + 1$. For $\alpha \in \mathbb{P}_N$ we use $X^{(\alpha)}$ to denote $\mathcal{X} \cap (\mathbb{P}_n \times \{\alpha\})$.

Lemma 1.2. \mathcal{X} is a nonsingular hypersurface of $\mathbb{P}_n \times \mathbb{P}_N$ of bidegree $(\delta, 1)$.

Proof. Take an arbitrary point (y, α) of \mathcal{X} with $y \in \mathbb{P}_n$ and $\alpha \in \mathbb{P}_N$. Choose a homogeneous coordinate system $[z_0, z_1, \dots, z_n]$ of \mathbb{P}_n so that

y is given by $[z_0, z_1, \dots, z_n] = [1, 0, \dots, 0]$. In other words, y is the origin in the inhomogeneous coordinate system

$$(x_1, \dots, x_n) = \left(\frac{z_1}{z_0}, \frac{z_2}{z_0}, \dots, \frac{z_n}{z_0} \right)$$

associated to the homogeneous coordinate system $[z_0, z_1, \dots, z_n]$.

The hypersurface \mathcal{X} in $\mathbb{P}_n \times \mathbb{P}_N$ is nonsingular if and only if its pullback $\tilde{\mathcal{X}}$ to $(\mathbb{C}^{N+1} - 0) \times (\mathbb{C}^{n+1} - 0)$ is nonsingular, because locally at points of \mathcal{X} the pullback $\tilde{\mathcal{X}}$ is simply equal to the product of \mathcal{X} with $(\mathbb{C} - 0) \times (\mathbb{C} - 0)$.

To determine whether $\tilde{\mathcal{X}}$ is nonsingular, we differentiate the defining function

$$f = \sum_{\nu_0 + \dots + \nu_n = \delta} \alpha_{\nu_0, \dots, \nu_n} z_0^{\nu_0} \dots z_n^{\nu_n}$$

with respect to each $\alpha_{\nu_0, \dots, \nu_n}$ and each z_j and evaluate the results at $z_0 = 1, z_1 = \dots = z_n = 0$ to see whether we get a nonzero $((N+1) + (n+1))$ -vector. We choose $\nu_0 = \delta, \nu_1 = \dots = \nu_n = 0$ and get

$$\left(\frac{\partial}{\partial \alpha_{\delta, 0, \dots, 0}} \right) f \Big|_{z_0=1, z_1=\dots=z_n=0} = 1$$

and conclude that the $((N+1) + (n+1))$ -vector is nonzero. Thus \mathcal{X} is nonsingular at every point $(y, \alpha) \in \mathbb{P}_n \times \mathbb{P}_N$ which belongs to \mathcal{X} . \square

Remark 1.3. Though \mathcal{X} is nonsingular at every point $(y, \alpha) \in \mathbb{P}_n \times \mathbb{P}_N$ which belongs to \mathcal{X} , the hypersurface $X^{(\alpha)}$ in \mathbb{P}_n which corresponds to α and is equal to $\mathcal{X} \cap (\mathbb{P}_n \times \{\alpha\})$ may have singularities.

Lemma 1.4. *Let ℓ be a positive integer and let L be a homogeneous polynomial of degree ℓ in the variables $\{\alpha_\nu\}_{|\nu|=\delta}$. Let $0 \leq p \neq q \leq n$ and $\nu, \mu \in (\mathbb{N} \cup \{0\})^{n+1}$ such that $\nu + e_q = \mu + e_p$. Then the $\mathcal{O}_{\mathbb{P}_n}(1) \times \mathcal{O}_{\mathbb{P}_N}(\ell - 1)$ -valued global holomorphic vector field*

$$L \left(z_q \left(\frac{\partial}{\partial \alpha_\nu} \right) - z_p \left(\frac{\partial}{\partial \alpha_\mu} \right) \right)$$

on $\mathbb{P}_n \times \mathbb{P}_N$ is tangential to \mathcal{X} .

Proof. The expression

$$L \left(z_q \left(\frac{\partial}{\partial \alpha_\nu} \right) - z_p \left(\frac{\partial}{\partial \alpha_\mu} \right) \right)$$

is a $\mathcal{O}_{\mathbb{P}_n}(1) \times \mathcal{O}_{\mathbb{P}_N}(\ell - 1)$ -valued global holomorphic vector field on $\mathbb{P}_n \times \mathbb{P}_N$, because the tangent bundle of \mathbb{P}_N is generated by global holomorphic vector fields of the form

$$\sum_{|\mu|=|\nu|=\delta} A_{\mu,\nu} \alpha_\mu \frac{\partial}{\partial \alpha_\nu}$$

with $A_{\mu,\nu} \in \mathbb{C}$.

The hypersurface \mathcal{X} is defined by $f = \sum_{|\nu|=\delta} \alpha_\nu z^\nu$. From

$$\frac{\partial f}{\partial \alpha_\nu} = z^\nu$$

and $\nu + e_q = \mu + e_p$ it follows that

$$\left(L \left(z_q \left(\frac{\partial}{\partial \alpha_\nu} \right) - z_p \left(\frac{\partial}{\partial \alpha_\mu} \right) \right) \right) f = L(z_q z^\nu - z_p z^\mu) = 0.$$

Hence the $\mathcal{O}_{\mathbb{P}_n}(1) \times \mathcal{O}_{\mathbb{P}_N}(\ell - 1)$ -valued global holomorphic vector field

$$L \left(z_q \left(\frac{\partial}{\partial \alpha_\nu} \right) - z_p \left(\frac{\partial}{\partial \alpha_\mu} \right) \right)$$

on $\mathbb{P}_n \times \mathbb{P}_N$ is tangential to the hypersurface \mathcal{X} of $\mathbb{P}_n \times \mathbb{P}_N$ □

Remark 1.5. The use of L is to make sure that we have a line-bundle-valued global holomorphic tangent vector field on $\mathbb{P}_n \times \mathbb{P}_N$. We will only use the case $\ell = 1$.

Lemma 1.6. *For any global holomorphic vector field ξ on \mathbb{P}_n there exists a global holomorphic vector field $\tilde{\xi}$ on $\mathbb{P}_n \times \mathbb{P}_N$ such that*

- (i) $\tilde{\xi}$ is tangential to \mathcal{X} , and
- (ii) $\tilde{\xi}$ is projected to ξ under the natural projection from $\mathbb{P}_n \times \mathbb{P}_N$ onto the second factor \mathbb{P}_n .

Proof. Consider the Euler sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}_n} \xrightarrow{\phi} \mathcal{O}_{\mathbb{P}_n}(1)^{\oplus(n+1)} \xrightarrow{\psi} T_{\mathbb{P}_n} \rightarrow 0,$$

where

$$\psi \left(\sum_{j=0}^n a_{j,0} z_j, \sum_{j=0}^n a_{j,1} z_j, \dots, \sum_{j=0}^n a_{j,n} z_j \right) = \sum_{j,k=0}^n a_{j,k} z_j \frac{\partial}{\partial z_k},$$

$$\phi(1) = (z_0, \dots, z_n).$$

Since $H^1(\mathbb{P}_n, \mathcal{O}_{\mathbb{P}_n})$ vanishes, it follows from the exact cohomology sequence of the Euler sequence that ξ is of the form $\sum_{j,k=0}^n a_{j,k} z_j \frac{\partial}{\partial z_k}$ for some complex numbers $a_{j,k}$.

For $0 \leq j, k \leq n$ with $j \neq k$ we define

$$\Phi_{j,k} : (\mathbb{N} \cup \{0\})^{n+1} \rightarrow (\mathbb{N} \cup \{0\})^{n+1}$$

as follows. For $\nu \in (\mathbb{N} \cup \{0\})^{n+1}$ we set

$$\begin{aligned} (\Phi_{j,k}(\nu))_\ell &= \nu_\ell \quad \forall \ell \neq j, k, \\ (\Phi_{j,k}(\nu))_j &= \nu_j - 1, \\ (\Phi_{j,k}(\nu))_k &= \nu_k + 1. \end{aligned}$$

For $\{\alpha_\nu\}_{\nu \in (\mathbb{N} \cup \{0\})^{n+1}}$ we define

$$\beta_\nu = - \sum_{0 \leq j, k \leq n, j \neq k} \alpha_{\Phi_{j,k}(\nu)} a_{j,k} (\nu_k + 1) - \sum_{j=0}^n \alpha_{\nu_0, \nu_1, \dots, \nu_n} a_{j,j} \nu_j.$$

Then for $f = \sum_{|\nu|=\delta} \alpha_\nu z^\nu$, we have

$$\xi(f) = \left(\sum_{j,k=0}^n a_{j,k} z_j \frac{\partial}{\partial z_k} \right) f = - \sum_{|\nu|=\delta} \beta_\nu z^\nu.$$

The verification is as follows. Since

$$\left(\frac{\partial}{\partial z_k} \right) f = \sum_{|\nu|=\delta} \alpha_\nu \nu_k z_0^{\nu_0} \cdots z_{k-1}^{\nu_{k-1}} z_k^{\nu_k-1} z_{k+1}^{\nu_{k+1}} \cdots z_n^{\nu_n},$$

it follows that

$$\begin{aligned} (1.6.1) \quad & \left(\sum_{j,k=0}^n a_{j,k} z_j \frac{\partial}{\partial z_k} \right) f \\ &= \sum_{j,k=0}^n a_{j,k} z_j \sum_{|\nu|=\delta} \alpha_\nu \nu_k z_0^{\nu_0} \cdots z_{k-1}^{\nu_{k-1}} z_k^{\nu_k-1} z_{k+1}^{\nu_{k+1}} \cdots z_n^{\nu_n}. \end{aligned}$$

So, the term on the right-hand side of (1.6.1) when $j = k$ is

$$a_{j,j} \alpha_{\nu_0, \nu_1, \dots, \nu_n} \nu_k z_0^{\nu_0} z_1^{\nu_1} \cdots z_n^{\nu_n}.$$

This means that the net effect is multiplication by ν_j when $j = k$. The contribution to the term on the right-hand side of (1.6.1) with $j \neq k$ is

$$a_{j,k} \nu_k \alpha_{\nu_0, \nu_1, \dots, \nu_n} z_0^{\nu_0} \cdots z_{j-1}^{\nu_{j-1}} z_j^{\nu_j+1} z_{j+1}^{\nu_{j+1}} \cdots z_{k-1}^{\nu_{k-1}} z_k^{\nu_k-1} z_{k+1}^{\nu_{k+1}} \cdots z_n^{\nu_n}.$$

We now change the dummy indices ν_j and ν_k to look at the coefficient of the monomial

$$z_0^{\nu_0} z_1^{\nu_1} \cdots z_n^{\nu_n}.$$

We change the dummy index ν_j to $\nu_j - 1$ and change the dummy index ν_k to $\nu_k + 1$ to get

$$(\nu_k + 1) a_{j,k} \alpha_{\nu_0, \dots, \nu_{j-1}, \nu_j-1, \nu_{j+1}, \dots, \nu_{k-1}, \nu_k+1, \nu_{k+1}, \dots, \nu_n} z_0^{\nu_0} z_1^{\nu_1} \cdots z_n^{\nu_n}.$$

This concludes the verification.

It now suffices to set

$$\tilde{\xi} = \sum_{j,k=0}^n a_{j,k} z_j \frac{\partial}{\partial z_k} + \sum_{|\nu|=\delta} \beta_\nu \frac{\partial}{\partial \alpha_\nu}.$$

□

Lemma 1.7. $T_{\mathcal{X}} \otimes \mathcal{O}_{\mathbb{P}_n}(1)$ is globally generated.

Proof. Take an arbitrary point (y, α) of \mathcal{X} with $y \in \mathbb{P}_n$ and $\alpha \in \mathbb{P}_N$. Again choose a homogeneous coordinate system $[z_0, z_1, \dots, z_n]$ of \mathbb{P}_n so that y is given by $[z_0, z_1, \dots, z_n] = [1, 0, \dots, 0]$. We choose a homogeneous linear polynomial L of the variables $\{\alpha_\nu\}_{|\nu|=\delta}$ such that $L(s) \neq 0$.

It is equivalent to look at the tangent bundle of the pullback $\tilde{\mathcal{X}}$ of $\mathcal{X} \subset (\mathbb{C}^{n+1} - 0) \times (\mathbb{C}^{N+1} - 0)$ under the natural projection $(\mathbb{C}^{n+1} - 0) \times (\mathbb{C}^{N+1} - 0) \rightarrow \mathbb{P}_n \times \mathbb{P}_N$. Take ν with $\nu_p > 0$ for some $1 \leq p \leq n$. Then there exists a unique $\mu \in (\mathbb{N} \cup \{0\})^{n+1}$ such that $\nu + e_0 = \mu + e_p$. At (y, α) the value of

$$L \left(z_0 \left(\frac{\partial}{\partial \alpha_\nu} \right) - z_p \left(\frac{\partial}{\partial \alpha_\mu} \right) \right)$$

is equal to $L(s) \left(\frac{\partial}{\partial \alpha_\nu} \right)$. Thus we conclude that the global holomorphic sections of $T_{\mathcal{X}} \times \mathcal{O}_{\mathbb{P}_n}(1)$ generate $\frac{\partial}{\partial \alpha_\nu}$ for $\nu \neq (\delta, 0, \dots, 0)$. By Lemma 1.6 the global holomorphic sections of $T_{\mathcal{X}} \times \mathcal{O}_{\mathbb{P}_n}(1)$ also generate $\frac{\partial}{\partial z_j}$ for $0 \leq j \leq n$. We thus conclude that global holomorphic sections of $T_{\mathcal{X}} \times \mathcal{O}_{\mathbb{P}_n}(1)$ generate a codimension 1 vector subspace of the tangent space of $\mathbb{P}_n \times \mathbb{P}_N$ at (y, α) . Since \mathcal{X} is nonsingular at (α, y) , it follows that global holomorphic sections of $T_{\mathcal{X}} \otimes \mathcal{O}_{\mathbb{P}_n}(1)$ generate $T_{\mathcal{X}} \otimes \mathcal{O}_{\mathbb{P}_n}(1)$. □

Lemma 1.8. Let q be a nonnegative integer. Global holomorphic sections of $T_{\mathcal{X}} \otimes \mathcal{O}_{\mathbb{P}_n}(q+1) \otimes \mathcal{O}_{\mathbb{P}_N}(q)$ generate all of its q -jets of \mathcal{X} .

Proof. Global holomorphic section of $\mathcal{O}_{\mathbb{P}_n}(q) \otimes \mathcal{O}_{\mathbb{P}_N}(q)$ generate all of its q -jets. Thus we can use the product of a global holomorphic section of $\mathcal{O}_{\mathbb{P}_n}(q) \otimes \mathcal{O}_{\mathbb{P}_N}(q)$ and a global holomorphic section of $T_{\mathcal{X}} \otimes \mathcal{O}_{\mathbb{P}_n}(1)$ to generate any prescribed q -jet of \mathcal{X} . □

1.9. Lie Derivatives

Let X be a complex manifold and ξ be a holomorphic vector field on X . Let $\varphi_{\xi,t}$ be a 1-parameter local biholomorphism defined by the vector field ξ so that

$$\left. \frac{\partial}{\partial t} \varphi_{\xi,t}^* g \right|_{t=0} = \xi(g)$$

for any local holomorphic function g on X . For any k -jet differential ω on X , we define the Lie derivative $\mathcal{L}ie_{\xi}(\omega)$ of ω with respect to ξ by

$$\mathcal{L}ie_{\xi}(\omega) = \left. \frac{\partial}{\partial t} \varphi_{X,t}^* \omega \right|_{t=0}.$$

Since

$$\begin{aligned} d \left(\left. \frac{\partial}{\partial t} \varphi_{X,t}^* \omega \right|_{t=0} \right) &= d \left(\lim_{t \rightarrow 0} \frac{1}{t} (\varphi_{X,t}^* \omega - \omega) \right) \\ &= \lim_{t \rightarrow 0} \frac{1}{t} (d\varphi_{X,t}^* \omega - d\omega) = \lim_{t \rightarrow 0} \frac{1}{t} (\varphi_{X,t}^* d\omega - d\omega) \\ &= \left. \frac{\partial}{\partial t} (\varphi_{X,t}^* (d\omega)) \right|_{t=0}, \end{aligned}$$

it follows that

$$d(\mathcal{L}ie_{\xi}(\omega)) = \mathcal{L}ie_{\xi}(d\omega).$$

Let η be a holomorphic ℓ -jet differential on X . The Leibniz product formula holds for the Lie derivatives of the product of ω and η so that

$$\mathcal{L}ie_{\xi}(\omega\eta) = \mathcal{L}ie_{\xi}(\omega)\eta + \omega\mathcal{L}ie_{\xi}(\eta).$$

Let (w_1, \dots, w_n) be a local coordinate system of X . Fix some $1 \leq i \leq n$. If $\omega = d^k w_i$ and $\xi = \sum_{j=1}^n g_j(w) \frac{\partial}{\partial w_j}$, then

$$\mathcal{L}ie_{\xi}(\omega) = d^k \left(\sum_{j=1}^n g_j(w) \left(\frac{\partial}{\partial w_j} \right) w_i \right) = d^k g_i(w).$$

Lemma 1.10. *Let k be a positive integer. Let X be a complex manifold and D be a complex hypersurface in X . Let ω be a holomorphic k -jet differential on X which vanishes at points of D to order p . Let ξ be a meromorphic vector field on X whose only possible poles are those of order at most q at D . If $p \geq q + k$, then $\mathcal{L}ie_{\xi}(\omega)$ is a holomorphic k -jet differential on X which vanishes at points of D to order $p - (q + k)$.*

Proof. Locally we can write

$$\omega = \sum_{\lambda_{1,1}, \dots, \lambda_{1,n}, \dots, \lambda_{k,1}, \dots, \lambda_{k,n} \geq 0} h_{\lambda_{1,1}, \dots, \lambda_{1,n}, \dots, \lambda_{k,1}, \dots, \lambda_{k,n}}(w) \prod_{1 \leq \ell \leq k, 1 \leq j \leq n} (d^{\ell} w_j)^{\lambda_{\ell,j}}$$

with $h_{\lambda_{1,1}, \dots, \lambda_{1,n}, \dots, \lambda_{k,1}, \dots, \lambda_{k,n}}(w)$ vanishing at points of D to order p .

Locally we can write $\xi = \sum_{j=1}^n g_j(w) \frac{\partial}{\partial w_j}$ with the pole order of g_j at most q at D . When we apply $\mathcal{L}ie_\xi$, by the Leibniz product rule we apply it only to one factor of each term separately and sum up. When it is applied to $h_{\lambda_{1,1}, \dots, \lambda_{1,n}, \dots, \lambda_{k,1}, \dots, \lambda_{k,n}}$, we end up with

$$\sum_{j=1}^n g_j(w) \frac{\partial}{\partial w_j} (h_{\lambda_{1,1}, \dots, \lambda_{1,n}, \dots, \lambda_{k,1}, \dots, \lambda_{k,n}}(w))$$

which vanishes at points of D to order $p - (q + 1) \geq p - (q + k)$. Since the pole order of $d^\ell g_j$ is at most $q + \ell$ at D , when it is applied to $d^\ell w_j$ and then multiplied by $h_{\lambda_{1,1}, \dots, \lambda_{1,n}, \dots, \lambda_{k,1}, \dots, \lambda_{k,n}}$, we end up with $h_{\lambda_{1,1}, \dots, \lambda_{1,n}, \dots, \lambda_{k,1}, \dots, \lambda_{k,n}} d^\ell g_j$ which vanishes at points of D to order $p - (q + \ell) \geq p - (q + k)$. \square

2. CONSTRUCTION OF SLANTED VECTOR FIELDS FOR JET SPACE

2.1. Binary Trees of Indices.

For a convenient description of vector fields on the total space of k -jets, we introduce now the binary trees of indices. A *binary tree of indices of order k* , which we denote by $\mathbf{p}^{(k)}$, is a collection

$$\left\{ p_{\gamma_1, \gamma_2, \dots, \gamma_j} \mid 1 \leq j \leq k, \text{ each of } \gamma_1, \dots, \gamma_j = 0, 1 \right\}$$

of indices, where each $p_{\gamma_1, \gamma_2, \dots, \gamma_j}$ is an integer satisfying $0 \leq p_{\gamma_1, \gamma_2, \dots, \gamma_j} \leq n$. We will use the interpretation of this collection of indices as a tree as follows. The binary tree starts with two nodes p_0, p_1 at its root and each of these two nodes p_0, p_1 branches out into a pair of nodes. On top of the node p_0 there are two nodes $p_{0,0}$ and $p_{0,1}$. On top of the node p_1 there are two nodes $p_{1,0}$ and $p_{1,1}$. Each of the four nodes $p_{0,0}, p_{0,1}, p_{1,0}, p_{1,1}$ again branches out into a pair of nodes. On top of the node p_{γ_1, γ_2} ($\gamma_1 = 0, 1; \gamma_2 = 0, 1$) there are two nodes $p_{\gamma_1, \gamma_2, 0}$ and $p_{\gamma_1, \gamma_2, 1}$. At the j -th branching into a pair of two nodes for each node, we have two nodes

$$p_{\gamma_1, \gamma_2, \dots, \gamma_j, 0}, p_{\gamma_1, \gamma_2, \dots, \gamma_j, 1}$$

on top of the node $p_{\gamma_1, \gamma_2, \dots, \gamma_j}$ for

$$\gamma_1 = 0, 1; \gamma_2 = 0, 1; \dots, \gamma_j = 0, 1.$$

At the top the tree, after the $(k - 1)$ -th branching we have the nodes

$$p_{\gamma_1, \gamma_2, \dots, \gamma_k} \quad (\gamma_1 = 0, 1; \gamma_2 = 0, 1; \dots, \gamma_k = 0, 1).$$

We will use the convention that the binary tree, as a collection of indices, will be denoted by a lower case Gothic letter and its indices

are denoted by the corresponding lower case Latin letter. When $k = 0$, we use the convention that $\mathbf{p}^{(0)}$ is just the empty set.

We now introduce the *truncation* of a binary tree of order k to form a binary subtree of order $k - j$. We denote by $\mathbf{p}^{(k; \tilde{\gamma}_1, \dots, \tilde{\gamma}_j)}$ the binary tree

$$\left\{ p_{\tilde{\gamma}_1, \tilde{\gamma}_2, \dots, \tilde{\gamma}_j, \gamma_{j+1}, \dots, \gamma_i} \mid j+1 \leq i \leq k, \text{ each of } \gamma_{j+1}, \dots, \gamma_i = 0, 1 \right\}$$

of order $k - j$. We call $\mathbf{p}^{(k; \tilde{\gamma}_1, \dots, \tilde{\gamma}_j)}$ the *truncation* of $\mathbf{p}^{(k)}$ at its node $p_{\tilde{\gamma}_1, \tilde{\gamma}_2, \dots, \tilde{\gamma}_j}$.

In this paper we work only with the following special kind of binary trees. A binary tree $\mathbf{p}^{(k)}$ of order k is said to *have level-wise homogeneous branches* if for every $1 \leq j < l$ and for any pairs

$$(\tilde{\gamma}_1, \tilde{\gamma}_2, \dots, \tilde{\gamma}_j) \quad \text{and} \quad (\hat{\gamma}_1, \hat{\gamma}_2, \dots, \hat{\gamma}_j)$$

of j -tuples of 0's and 1's, the two truncations

$$\mathbf{p}^{(k; \tilde{\gamma}_1, \tilde{\gamma}_2, \dots, \tilde{\gamma}_j)}, \quad \mathbf{p}^{(k; \hat{\gamma}_1, \hat{\gamma}_2, \dots, \hat{\gamma}_j)}$$

of $\mathbf{p}^{(k)}$ are identical binary trees of order $k - j$.

Lemma 2.2. *Let*

$$\xi_j^{(i)} = d^i \log z_j \quad (i \geq 1, \quad 0 \leq j \leq n)$$

and

$$\xi^{(i)} = \left(\xi_0^{(i)}, \xi_1^{(i)}, \dots, \xi_n^{(i)} \right).$$

Let $f = \sum_{\nu} \alpha_{\nu} z^{\nu}$ be a polynomial of homogeneous degree δ . Let $\Phi_{\nu}^{(0)} \equiv 1$ and inductively for $k \geq 0$,

(2.2.1) _{$k+1$}

$$\Phi_{\nu}^{(k+1)}(\xi^{(1)}, \dots, \xi^{(k+1)}) = \left(\sum_{\ell=0}^n \nu_{\ell} \xi_{\ell}^{(1)} \right) \Phi_{\nu}^{(k)} + \sum_{\ell=0}^n \sum_{i=1}^k \xi_{\ell}^{(i+1)} \frac{\partial \Phi_{\nu}^{(k)}}{\partial \xi_{\ell}^{(i)}}.$$

Then

$$(2.2.2) \quad d^k f = \sum_{\nu} \alpha_{\nu} \Phi_{\nu}^{(k)}(\xi^{(1)}, \dots, \xi^{(k)}) z^{\nu},$$

where for the differentiation α_{ν} is regarded as a constant and $\Phi_{\nu}^{(k)}$ is of homogeneous weight k when $\xi_{\ell}^{(j)}$ is given the weight j and is independent of (z_0, \dots, z_n) . Moreover, the coefficients of $\Phi_{\nu}^{(k)}$ in $\xi^{(1)}, \dots, \xi^{(k)}$ are

polynomials in $\nu = (\nu_0, \dots, \nu_n)$ of degree at most k with universal coefficients. As a polynomial in $\nu = (\nu_0, \dots, \nu_n)$, the degree of

$$\Phi_\nu^{(k)} - \left(\sum_{\ell=0}^n \nu_\ell \xi_\ell^{(1)} \right)^k$$

is at most $k - 1$.

Proof. To prove the Lemma, we define $\Phi_\nu^{(k)}$ by (2.2.2) and verify (2.2.1)₁ and (2.2.1)_{k+1} for $k \geq 1$. The verification is as follows. Clearly,

$$\Phi_\nu^{(1)} = \sum_{\ell=0}^n \nu_\ell \xi_\ell^{(1)}$$

and is homogeneous of weight 1. To verify (2.2.2) when k is replaced by $k + 1$, we apply d to both sides of (2.2.2). The effect of applying d to z^ν is to replace the factor z^ν by $\Phi_\nu^{(1)}$ which is $\sum_{\ell=0}^n \nu_\ell \xi_\ell^{(1)}$. The effect of applying d to the other factor $\Phi_\nu^{(k)}(\xi^{(1)}, \dots, \xi^{(k)})$ is

$$\sum_{\ell=0}^n \sum_{i=1}^k \xi_\ell^{(i+1)} \frac{\partial \Phi_\nu^{(k)}}{\partial \xi_\ell^{(i)}}$$

by the chain rule. We now consider the question of weights and homogeneous degrees. For $k \geq 1$ the first term on the right-hand side of (2.2.1)_{k+1} is the product of the factors $\Phi_\nu^{(1)} \Phi_\nu^{(k)}$ which are respectively homogeneous of weights 1 and k by induction hypothesis. The second term on the right-hand side of (2.2.1)_{k+1} is the sum of a product of two factors

$$\xi^{(i+1)} \frac{\partial \Phi_\nu^{(k)}}{\partial \xi_\ell^{(i)}}$$

which are respectively homogeneous of weights $i + 1$ and $k - i$ by induction hypothesis. This finishes the verification of (2.2.1)_{k+1} and the homogeneity of $\Phi_\nu^{(k+1)}$ of weight $k + 1$. From (2.2.1)_{k+1} it is clear by induction on k that the coefficients of $\Phi_\nu^{(k)}$ in $\xi^{(1)}, \dots, \xi^{(k)}$ are polynomials in ν of degree at most k with universal coefficients.

Finally, in (2.2.1)_{k+1} the term

$$\sum_{\ell=0}^n \sum_{i=1}^k \xi_\ell^{(i+1)} \frac{\partial \Phi_\nu^{(k)}}{\partial \xi_\ell^{(i)}}$$

on the right-hand side as a polynomial in ν is of degree no higher than that of $\Phi_\nu^{(k)}$ which is no higher than k . Thus for the induction process

of going from Step k to Step $k + 1$, if the degree of

$$\Phi_\nu^{(k)} - \left(\sum_{\ell=0}^n \nu_\ell \xi_\ell^{(1)} \right)^k$$

is at most $k - 1$ in $\nu = (\nu_0, \dots, \nu_n)$, then by $(2.2.1)_{k+1}$ the degree of

$$\Phi_\nu^{(k+1)} - \left(\sum_{\ell=0}^n \nu_\ell \xi_\ell^{(1)} \right)^{k+1}$$

is at most k in $\nu = (\nu_0, \dots, \nu_n)$. \square

2.3. Construction by Induction.

Let $\mathbf{p}^{(k)}$ be a binary tree of indices of order k . We denote by $\lambda^{(k)}$ a multi-index of $n + 1$ components with total degree $\delta - k$. For $1 \leq j \leq k$ and for the choice of each $\gamma_1, \dots, \gamma_j$ being 0 or 1, we denote by $\lambda^{(k; \gamma_1, \dots, \gamma_j)}$ the multi-index

$$\lambda^{(k)} + \sum_{i=1}^j e_{p_{\gamma_1, \dots, \gamma_i}}$$

with total degree $|\lambda^{(k; \gamma_1, \dots, \gamma_j)}| = \delta - k + j$. Recall that $e_{p_{\gamma_1, \dots, \gamma_i}}$ is the index of $n + 1$ components whose only nonzero component is the $p_{\gamma_1, \dots, \gamma_i}$ -th component which is 1.

For $0 \leq k \leq n - 1$, for any multi-index $\lambda^{(k)}$ of $n + 1$ components with total degree $\delta - k$, and for any binary tree $\mathbf{p}^{(k)}$ of order k which has level-wise homogeneous branches, we are going to explicitly construct by induction on k ,

$$\Theta_{\lambda^{(k)}, \mathbf{p}^{(k)}}^{(j)}, \Psi_{\lambda^{(k)}, \mathbf{p}^{(k)}}^{(j)} \quad (k \leq j \leq n)$$

such that

(1) $\Psi_{\lambda^{(k)}, \mathbf{p}^{(k)}}^{(j)}$ is a rational function of the entries of $\lambda^{(k)}$ and $\xi_q^{(\ell)}$ with q equal to some $p_{\gamma_1, \gamma_2, \dots, \gamma_\ell}$ for $1 \leq \ell \leq k$,

(2) $\Theta_{\lambda^{(k)}, \mathbf{p}^{(k)}}^{(k)}$ is a meromorphic vector field on the parameter space with coordinates α_ν (for multi-indices ν of $n + 1$ components of total degree δ) which is a linear combination of $\frac{\partial}{\partial \alpha_\nu}$ (for $|\nu| = \delta$) whose coefficients are rational functions of

$$z_0, \dots, z_n, \xi_\ell^{(j)} \quad (1 \leq j \leq k, 0 \leq \ell \leq n)$$

and which satisfies

$$\Theta_{\lambda^{(k)}, \mathbf{p}^{(k)}}^{(k)} (d^j f) = 0 \quad \text{for } 0 \leq j \leq k - 1,$$

$$\Theta_{\lambda^{(k)}, \mathbf{p}^{(k)}}^{(k)} (d^j f) = z^{\lambda^{(k)}} \Psi_{\lambda^{(k)}, \mathbf{p}^{(k)}}^{(j)} \quad \text{for } k \leq j \leq n-1.$$

Here we regard $\Theta_{\lambda^{(k)}, \mathbf{p}^{(k)}}^{(k)}$ as a vector field on the space with variables α_ν for $|\nu| = \delta$ while the variables z_0, \dots, z_n and $\xi_\ell^{(j)}$ ($1 \leq j \leq k, 0 \leq \ell \leq n$) are regarded as constants. It is the same as regarding $\Theta_{\lambda^{(k)}, \mathbf{p}^{(k)}}^{(k)}$ as a vector field on the space with variables α_ν for $|\nu| = \delta$ and the variables z_0, \dots, z_n and $\xi_\ell^{(j)}$ ($1 \leq j \leq k, 0 \leq \ell \leq n$) when the coefficients for $\frac{\partial}{\partial z_\ell}$ and for $\frac{\partial}{\partial \xi_\ell^{(j)}}$ are all 0 for $1 \leq j \leq k, 0 \leq \ell \leq n$. The construction is as follows.

For $k = 0$ with the convention that $\mathbf{p}^{(0)}$ is the empty set, we simply set

$$\Theta_{\lambda^{(0)}, \mathbf{p}^{(0)}}^{(0)} = \frac{\partial}{\partial \alpha_{\lambda^{(0)}}}$$

and

$$(2.3.1) \quad \Psi_{\lambda^{(0)}, \mathbf{p}^{(0)}}^{(j)} = \Phi_{\lambda^{(0)}}^{(j)}$$

for $0 \leq j \leq n-1$. It is clear that $\Psi_{\lambda^{(0)}, \mathbf{p}^{(0)}}^{(j)}$ is of homogeneous weight j in $\xi_t^{(\ell)}$ ($1 \leq \ell \leq j, 0 \leq t \leq n$) and is independent of (z_0, \dots, z_n) and is a polynomial in the $n+1$ components of $\lambda^{(0)}$ of degree $\leq j$. Moreover, it follows from $f = \sum_\nu \alpha_\nu z^\nu$ and the definition of $\Phi_\nu^{(j)}$ that

$$(2.3.2) \quad \Theta_{\lambda^{(0)}, \mathbf{p}^{(0)}}^{(0)} (d^j f) = z^{\lambda^{(0)}} \Psi_{\lambda^{(0)}, \mathbf{p}^{(0)}}^{(j)}$$

for $0 \leq j \leq n-1$. Suppose the construction has been done for the step k and we are going to construct for the step $k+1$. Define

$$\Theta_{\lambda^{(k+1)}, \mathbf{p}^{(k+1)}}^{(k+1)} = \frac{\Theta_{\lambda^{(k+1); p_0}, \mathbf{p}^{(k+1); p_0}}^{(k)}}{z_{p_0} \Psi_{\lambda^{(k+1); p_0}, \mathbf{p}^{(k+1); p_0}}^{(k)}} - \frac{\Theta_{\lambda^{(k+1); p_1}, \mathbf{p}^{(k+1); p_1}}^{(k)}}{z_{p_1} \Psi_{\lambda^{(k+1); p_1}, \mathbf{p}^{(k+1); p_1}}^{(k)}}$$

and

$$(2.3.3) \quad \Psi_{\lambda^{(k+1)}, \mathbf{p}^{(k+1)}}^{(j)} = \frac{\Psi_{\lambda^{(k+1); p_0}, \mathbf{p}^{(k+1); p_0}}^{(j)}}{\Psi_{\lambda^{(k+1); p_0}, \mathbf{p}^{(k+1); p_0}}^{(k)}} - \frac{\Psi_{\lambda^{(k+1); p_1}, \mathbf{p}^{(k+1); p_1}}^{(j)}}{\Psi_{\lambda^{(k+1); p_1}, \mathbf{p}^{(k+1); p_1}}^{(k)}}$$

for $j \geq k+1$.

Lemma 2.4. *For any integer k with $0 \leq k \leq \delta$ the following two identities hold.*

$$\begin{aligned} \Theta_{\lambda^{(k)}, \mathbf{p}^{(k)}}^{(k)} (d^j f) &= 0 \quad \text{for } 0 \leq j \leq k-1, \\ \Theta_{\lambda^{(k)}, \mathbf{p}^{(k)}}^{(k)} (d^j f) &= z^{\lambda^{(k)}} \Psi_{\lambda^{(k)}, \mathbf{p}^{(k)}}^{(j)} \quad \text{for } k \leq j \leq n. \end{aligned}$$

Proof. We prove by induction on $k \geq 0$. Since the positive integer δ is fixed once for all, the induction on $k \geq 0$ is the same as descending induction on the total degree $\delta - k$ of $\lambda^{(k)}$. In the case $k = 0$ the statement is simply (2.3.2). To go from Step k to Step $k + 1$, we have

$$\Theta_{\lambda^{(k+1)}, \mathbf{p}^{(k+1)}}^{(k+1)} (d^j f) = 0 \quad \text{for } 0 \leq j \leq k - 1,$$

because

$$\Theta_{\lambda^{(k)}, \mathbf{p}^{(k)}}^{(k)} (d^j f) = 0 \quad \text{for } 0 \leq j \leq k - 1$$

and

$$\Theta_{\lambda^{(k+1)}, \mathbf{p}^{(k+1)}}^{(k+1)} = \frac{\Theta_{\lambda^{(k+1); p_0}, \mathbf{p}^{(k+1); p_0}}^{(k)}}{z_{p_0} \Psi_{\lambda^{(k+1); p_0}, \mathbf{p}^{(k+1); p_0}}^{(k)}} - \frac{\Theta_{\lambda^{(k+1); p_1}, \mathbf{p}^{(k+1); p_1}}^{(k)}}{z_{p_1} \Psi_{\lambda^{(k+1); p_1}, \mathbf{p}^{(k+1); p_1}}^{(k)}}.$$

We have

$$\Theta_{\lambda^{(k+1)}, \mathbf{p}^{(k+1)}}^{(k+1)} (d^k f) = 0,$$

because

$$\begin{aligned} & \Theta_{\lambda^{(k+1)}, \mathbf{p}^{(k+1)}}^{(k+1)} (d^k f) \\ &= \frac{\Theta_{\lambda^{(k+1); p_0}, \mathbf{p}^{(k+1); p_0}}^{(k)} (d^k f)}{z_{p_0} \Psi_{\lambda^{(k+1); p_0}, \mathbf{p}^{(k+1); p_0}}^{(k)}} - \frac{\Theta_{\lambda^{(k+1); p_1}, \mathbf{p}^{(k+1); p_1}}^{(k)} (d^k f)}{z_{p_1} \Psi_{\lambda^{(k+1); p_1}, \mathbf{p}^{(k+1); p_1}}^{(k)}} \\ &= \frac{z^{\lambda^{(k+1); p_0}} \Psi_{\lambda^{(k+1); p_0}, \mathbf{p}^{(k+1); p_0}}^{(k+1)}}{z_{p_0} \Psi_{\lambda^{(k+1); p_0}, \mathbf{p}^{(k+1); p_0}}^{(k)}} - \frac{z^{\lambda^{(k+1); p_1}} \Psi_{\lambda^{(k+1); p_1}, \mathbf{p}^{(k+1); p_1}}^{(j)}}{z_{p_1} \Psi_{\lambda^{(k+1); p_1}, \mathbf{p}^{(k+1); p_1}}^{(k)}} \\ &= \lambda^{(k+1)} - \lambda^{(k+1)} = 0. \end{aligned}$$

Moreover, for $k + 1 \leq j \leq n - 1$,

$$\begin{aligned} & \Theta_{\lambda^{(k+1)}, \mathbf{p}^{(k+1)}}^{(k+1)} (d^j f) \\ &= \frac{\Theta_{\lambda^{(k+1); p_0}, \mathbf{p}^{(k+1); p_0}}^{(k)} (d^j f)}{z_{p_0} \Psi_{\lambda^{(k+1); p_0}, \mathbf{p}^{(k+1); p_0}}^{(k)}} - \frac{\Theta_{\lambda^{(k+1); p_1}, \mathbf{p}^{(k+1); p_1}}^{(k)} (d^j f)}{z_{p_1} \Psi_{\lambda^{(k+1); p_1}, \mathbf{p}^{(k+1); p_1}}^{(k)}} \\ &= \frac{z^{\lambda^{(k+1); p_0}} \Psi_{\lambda^{(k+1); p_0}, \mathbf{p}^{(k+1); p_0}}^{(j)}}{z_{p_0} \Psi_{\lambda^{(k+1); p_0}, \mathbf{p}^{(k+1); p_0}}^{(k)}} - \frac{z^{\lambda^{(k+1); p_1}} \Psi_{\lambda^{(k+1); p_1}, \mathbf{p}^{(k+1); p_1}}^{(j)}}{z_{p_1} \Psi_{\lambda^{(k+1); p_1}, \mathbf{p}^{(k+1); p_1}}^{(k)}} \\ &= z^{\lambda^{(k+1)}} \left(\frac{\Psi_{\lambda^{(k+1); p_0}, \mathbf{p}^{(k+1); p_0}}^{(j)}}{\Psi_{\lambda^{(k+1); p_0}, \mathbf{p}^{(k+1); p_0}}^{(k)}} - \frac{\Psi_{\lambda^{(k+1); p_1}, \mathbf{p}^{(k+1); p_1}}^{(j)}}{\Psi_{\lambda^{(k+1); p_1}, \mathbf{p}^{(k+1); p_1}}^{(k)}} \right) \\ &= z^{\lambda^{(k+1)}} \Psi_{\lambda^{(k+1)}, \mathbf{p}^{(k+1)}}^{(j)}. \end{aligned}$$

□

Lemma 2.5. *For $k \leq j \leq n - 1$ the function $\Psi_{\lambda^{(k)}, \mathbf{p}^{(k)}}^{(j)}$ is homogeneous of weight $j - k + 1$ in the variables $\xi_q^{(\ell)}$ ($0 \leq q \leq n$, $1 \leq \ell \leq n$) when $\xi_q^{(\ell)}$ is assigned the weight ℓ . Moreover, for $k \leq j \leq n$, as a function of the $n + 1$ components of the multi-index $\lambda^{(k)}$, the function $\Psi_{\lambda^{(k)}, \mathbf{p}^{(k)}}^{(j)}$ is a polynomial of degree at most $j - k$. In particular, $\Psi_{\lambda^{(k)}, \mathbf{p}^{(k)}}^{(k)}$ is independent of the multi-index $\lambda^{(k)}$ and*

$$\Psi_{\lambda^{(k)}, \mathbf{p}^{(k)}}^{(j)} = \frac{1}{\Psi_{\lambda^{(k;p_0)}, \mathbf{p}^{(k;p_0)}}^{(k-1)}} \left[\Psi_{\lambda^{(k;p_0)}, \mathbf{p}^{(k;p_0)}}^{(j)} - \Psi_{\lambda^{(k;p_1)}, \mathbf{p}^{(k;p_1)}}^{(j)} \right].$$

Proof. We prove the Lemma by induction on $0 \leq k \leq n - 1$.

First we look at the weight of $\Psi_{\lambda^{(k)}, \mathbf{p}^{(k)}}^{(j)}$ and show that $\Psi_{\lambda^{(k)}, \mathbf{p}^{(k)}}^{(j)}$ is homogeneous of weight $j - k + 1$ in the variables $\xi_q^{(\ell)}$ ($0 \leq q \leq n$, $1 \leq \ell \leq n$). Again, since the positive integer δ is fixed once for all, the induction on $0 \leq k \leq n - 1$ is the same as the descending induction on the total degree $\delta - k$ of $\lambda^{(k)}$. For $k = 0$ the conclusion clearly follows from (2.3.1). The derivation of Step $k + 1$ from Step k simply follows from (2.3.3), because, for $\gamma_1 = 0, 1$, with the weight of $\Psi_{\lambda^{(k+1;p_{\gamma_1})}, \mathbf{p}^{(k+1;p_{\gamma_1})}}^{(j)}$ being $j - k$ and the weight of $\Psi_{\lambda^{(k+1;p_{\gamma_1})}, \mathbf{p}^{(k+1;p_{\gamma_1})}}^{(k)}$ being 1, the weight

$$\frac{\Psi_{\lambda^{(k+1;p_{\gamma_1})}, \mathbf{p}^{(k+1;p_{\gamma_1})}}^{(j)}}{\Psi_{\lambda^{(k+1;p_{\gamma_1})}, \mathbf{p}^{(k+1;p_{\gamma_1})}}^{(k)}}$$

is equal to $j - (k + 1)$.

Now we show that $\Psi_{\lambda^{(k)}, \mathbf{p}^{(k)}}^{(k)}$ is a polynomial of degree no more than $j - k$ in the $n + 1$ components of $\lambda^{(k)}$. Again for $k = 0$ the conclusion clearly follows from (2.3.1). For the derivation of Step $k + 1$ from Step k , since the total degree of $\lambda^{(k+1;p_{\gamma_1})}$ is $\delta - (k + 1) - 1 = \delta - k$, it follows from Step k that $\Psi_{\lambda^{(k+1;p_{\gamma_1})}, \mathbf{p}^{(k+1;p_{\gamma_1})}}^{(j)}$ is a polynomial of degree no more than zero in the $n + 1$ components of $\lambda^{(k+1;p_{\gamma_1})}$ and is therefore independent of $\lambda^{(k+1;p_{\gamma_1})}$ for $\gamma_1 = 0, 1$. Since the binary tree $\mathbf{p}^{(k+1)}$ is assumed to have level-wise homogeneous branches (see the paragraph preceding Lemma(2.2)), it follows that the two truncations $\mathbf{p}^{(k+1;0)}$ and $\mathbf{p}^{(k+1;1)}$ are identical binary trees and

$$\Psi_{\lambda^{(k+1;p_0)}, \mathbf{p}^{(k+1;p_0)}}^{(k)} = \Psi_{\lambda^{(k+1;p_1)}, \mathbf{p}^{(k+1;p_1)}}^{(k)}.$$

Hence by (2.3.3),

$$\begin{aligned} \Psi_{\lambda^{(k+1)}, \mathbf{p}^{(k+1)}}^{(j)} &= \frac{\Psi_{\lambda^{(k+1;p_0)}, \mathbf{p}^{(k+1;p_0)}}^{(j)}}{\Psi_{\lambda^{(k+1;p_0)}, \mathbf{p}^{(k+1;p_0)}}^{(k)}} - \frac{\Psi_{\lambda^{(k+1;p_1)}, \mathbf{p}^{(k+1;p_1)}}^{(j)}}{\Psi_{\lambda^{(k+1;p_1)}, \mathbf{p}^{(k+1;p_1)}}^{(k)}} \\ &= \frac{1}{\Psi_{\lambda^{(k+1;p_0)}, \mathbf{p}^{(k+1;p_0)}}^{(k)}} \left[\Psi_{\lambda^{(k+1;p_0)}, \mathbf{p}^{(k+1;p_0)}}^{(j)} - \Psi_{\lambda^{(k+1;p_1)}, \mathbf{p}^{(k+1;p_1)}}^{(j)} \right]. \end{aligned}$$

By induction assumption $\Psi_{\lambda^{(k+1;p_0)}, \mathbf{p}^{(k+1;p_0)}}^{(k)}$ is a polynomial in the $n+1$ components of the multi-index $\lambda^{(k+1;p_0)}$ of degree $j-k$ and $\Psi_{\lambda^{(k+1;p_1)}, \mathbf{p}^{(k+1;p_1)}}^{(k)}$ is a polynomial of the $n+1$ components of the multi-index $\lambda^{(k+1;p_1)}$ of degree at most $j-k$. Now

$$\lambda^{(k+1;p_0)} - \lambda^{(k+1;p_1)} = e_{p_0} - e_{p_1}$$

which means that the multi-index $\lambda^{(k+1;p_0)}$ is a translate of the multi-index $\lambda^{(k+1;p_1)}$ by the multi-index $e_{p_0} - e_{p_1}$ and is independent of the $n+1$ components of the multi-index $\lambda^{(k+1)}$. Thus the difference

$$\Psi_{\lambda^{(k+1;p_0)}, \mathbf{p}^{(k+1;p_0)}}^{(j)} - \Psi_{\lambda^{(k+1;p_1)}, \mathbf{p}^{(k+1;p_1)}}^{(j)}$$

is a polynomial of the $n+1$ components of the multi-index $\lambda^{(k+1)}$ of degree at most $j-k-1$. The last statement follows from Lemma 2.4. \square

Lemma 2.6. *If $\lambda^{(k)}$ is a multi-index of $n+1$ components with total degree $\delta-k$ and $\mathbf{p}^{(k)}$ is any binary tree of order k which has level-wise homogeneous branching, then*

$$\Psi_{\lambda^{(k)}, \mathbf{p}^{(k)}}^{(k)} = k \left(\xi_{p_0}^{(1)} - \xi_{p_1}^{(1)} \right),$$

where the nodes of $\mathbf{p}^{(k)}$ are denoted by $p_{\gamma_1, \dots, \gamma_j}$ with $1 \leq j \leq k$ and each γ_ℓ taking on the value 0 or 1 for $1 \leq \ell \leq j$. As a consequence,

$$\Psi_{\lambda^{(k)}, \mathbf{p}^{(k)}}^{(j)} = \frac{1}{k \left(\xi_{p_0}^{(1)} - \xi_{p_1}^{(1)} \right)} \left[\Psi_{\lambda^{(k;p_0)}, \mathbf{p}^{(k;p_0)}}^{(j)} - \Psi_{\lambda^{(k;p_1)}, \mathbf{p}^{(k;p_1)}}^{(j)} \right].$$

Proof. First we make the following simple observation. Let $G(\nu_0, \dots, \nu_n)$ be a polynomial in ν_0, \dots, ν_n of degree no more than M . For any $p \neq q$ define

$$\begin{aligned} (\Delta_{p,q} G)(\nu_0, \dots, \nu_n) &= G(\nu_0, \dots, \nu_{p-1}, \nu_p + 1, \nu_{p+1}, \dots, \nu_n) \\ &\quad - G(\nu_0, \dots, \nu_{q-1}, \nu_q + 1, \nu_{q+1}, \dots, \nu_n). \end{aligned}$$

Then $(\Delta_{p,q} G)(\nu_0, \dots, \nu_n)$ is a polynomial in ν_0, \dots, ν_n of degree no more than $M-1$, because we can write

$$(\Delta_{p,q} G)(\nu_0, \dots, \nu_n)$$

$$\begin{aligned}
&= [G(\nu_0, \dots, \nu_{p-1}, \nu_p + 1, \nu_{p+1}, \dots, \nu_n) - G(\nu_0, \dots, \nu_n)] \\
&\quad - [G(\nu_0, \dots, \nu_{q-1}, \nu_q + 1, \nu_{q+1}, \dots, \nu_n) - G(\nu_0, \dots, \nu_n)]
\end{aligned}$$

and clearly each of the two terms

$$G(\nu_0, \dots, \nu_{p-1}, \nu_p + 1, \nu_{p+1}, \dots, \nu_n) - G(\nu_0, \dots, \nu_n)$$

and

$$G(\nu_0, \dots, \nu_{q-1}, \nu_q + 1, \nu_{q+1}, \dots, \nu_n) - G(\nu_0, \dots, \nu_n)$$

is a polynomial in ν_0, \dots, ν_n of degree no more than $M - 1$. As a consequence,

(2.6.1) if

$$r_1 \neq s_1, \dots, r_{M+1} \neq s_{M+1},$$

then

$$\Delta_{r_1, s_1} \cdots \Delta_{r_M, s_M} G$$

is of degree zero in ν_0, \dots, ν_n and $\Delta_{r_1, s_1} \cdots \Delta_{r_{N+1}, s_{N+1}} G$ is identically zero for any polynomial $G(\nu_0, \dots, \nu_n)$ in ν_0, \dots, ν_n of degree no more than N .

Let $r_j = p_{\gamma_1, \dots, \gamma_{j-1}, 0}$ and $s_j = p_{\gamma_1, \dots, \gamma_{j-1}, 1}$. Since the binary tree $\mathbf{p}^{(k)}$ of order k has level-wise homogeneous branches, the values of $r_j = p_{\gamma_1, \dots, \gamma_{j-1}, 0}$ and $s_j = p_{\gamma_1, \dots, \gamma_{j-1}, 1}$ are independent of the choices of the values 0 or 1 for $\gamma_1, \dots, \gamma_{j-1}$. Let $\mathbf{p}^{(k-j)} = \mathbf{p}^{(k; \gamma_1, \dots, \gamma_{j-1})}$. Again we know that $\mathbf{p}^{(k-j)}$ is independent of the choices of the values 0 or 1 for $\gamma_1, \dots, \gamma_{j-1}$ because the binary tree $\mathbf{p}^{(k)}$ of order k has level-wise homogeneous branches. By the last statement of Lemma 2.5, we have

$$\begin{aligned}
(2.6.2) \quad \Psi_{\lambda^{(k)}, \mathbf{p}^{(k)}}^{(j)} &= \frac{1}{\Psi_{\lambda^{(k; p_0)}, \mathbf{p}^{(k-1)}}^{(k-1)}} \left[\Psi_{\lambda^{(k; p_0)}, \mathbf{p}^{(k; p_0)}}^{(j)} - \Psi_{\lambda^{(k; p_1)}, \mathbf{p}^{(k; p_1)}}^{(j)} \right] \\
&= \frac{1}{\Psi_{\mathbf{p}^{(k-1)}}^{(k-1)}} \left[\Psi_{\lambda^{(k; p_0)}, \mathbf{p}^{(k-1)}}^{(j)} - \Psi_{\lambda^{(k; p_1)}, \mathbf{p}^{(k-1)}}^{(j)} \right]
\end{aligned}$$

for any multi-index $\lambda^{(k)}$ of $n + 1$ components and total degree $\delta - k$. Here, because of the independence of $\Psi_{\lambda^{(k; p_0)}, \mathbf{p}^{(k; p_0)}}^{(k-1)}$ of the $n + 1$ components of the multi-index $\lambda^{(k; p_0)}$ by Lemma 2.5, we drop $\lambda^{(k; p_0)}$ from the subscript of $\Psi_{\lambda^{(k; p_0)}, \mathbf{p}^{(k; p_0)}}^{(k-1)}$ and simply write $\Psi_{\lambda^{(k; p_0)}, \mathbf{p}^{(k; p_0)}}^{(k-1)}$ as $\Psi_{\mathbf{p}^{(k; p_0)}}^{(k-1)}$. From

$$\Psi_{\lambda^{(0)}, \mathbf{p}^{(0)}}^{(j)} = \Phi_{\lambda^{(0)}}^{(j)}$$

in the formula (2.3.1) and from (2.6.2) it follows that

$$\Psi_{\lambda^{(k);\gamma_1,\dots,\gamma_{k-1}},\mathbf{p}^{(1)}}^{(j)} = \left[\frac{1}{\Psi_{\mathbf{p}^{(0)}}^{(0)}} \Delta_{r_k,s_k} \Phi_{\nu}^{(j)} \right]_{\nu=\lambda^{(k);\gamma_1,\dots,\gamma_{k-1}}}$$

for any $\lambda^{(k)}$ of $n+1$ components and total degree $\delta - k$, because the total degree of the multi-index $\lambda^{(k);\gamma_1,\dots,\gamma_{k-1}}$ is $\delta - 1$, which corresponds to the situation of $k = 1$ in (2.6.2). Inductively for $1 \leq \ell \leq k$ we are going to verify that

$$(2.6.3) \quad \Psi_{\lambda^{(k);\gamma_1,\dots,\gamma_{k-\ell}},\mathbf{p}^{(\ell)}}^{(j)} = \left[\frac{1}{\Psi_{\mathbf{p}^{(0)}}^{(0)} \cdots \Psi_{\mathbf{p}^{(\ell-1)}}^{(\ell-1)}} \Delta_{r_{k-\ell+1},s_{k-\ell+1}} \cdots \Delta_{r_k,s_k} \Phi_{\nu}^{(j)} \right]_{\nu=\lambda^{(k);\gamma_1,\dots,\gamma_{k-\ell}}}$$

for any $\lambda^{(k)}$ of $n+1$ components and total degree $\delta - k$. To go from Step ℓ to Step $\ell+1$, by (2.6.2) we have

$$\begin{aligned} & \Psi_{\lambda^{(k);\gamma_1,\dots,\gamma_{k-\ell-1}},\mathbf{p}^{(\ell+1)}}^{(j)} \\ &= \frac{1}{\Psi_{\mathbf{p}^{(\ell)}}^{(\ell)}} \left\{ \Psi_{\lambda^{(k);\gamma_1,\dots,\gamma_{k-\ell-1},0},\mathbf{p}^{(\ell)}}^{(j)} - \Psi_{\lambda^{(k);\gamma_1,\dots,\gamma_{k-\ell-1},1},\mathbf{p}^{(\ell)}}^{(j)} \right\} \\ &= \frac{1}{\Psi_{\mathbf{p}^{(\ell)}}^{(\ell)}} \left\{ \Psi_{\lambda^{(k);\gamma_1,\dots,\gamma_{k-\ell-1},0},\mathbf{p}^{(\ell)}}^{(j)} - \Psi_{\lambda^{(k);\gamma_1,\dots,\gamma_{k-\ell-1},1},\mathbf{p}^{(\ell)}}^{(j)} \right\} \\ &= \frac{1}{\Psi_{\mathbf{p}^{(\ell)}}^{(\ell)}} \left\{ \left[\frac{1}{\Psi_{\mathbf{p}^{(0)}}^{(0)} \cdots \Psi_{\mathbf{p}^{(\ell-1)}}^{(\ell-1)}} \Delta_{r_{k-\ell+1},s_{k-\ell+1}} \cdots \Delta_{r_k,s_k} \Phi_{\nu}^{(j)} \right]_{\nu=\lambda^{(k);\gamma_1,\dots,\gamma_{k-\ell-1},0}} \right. \\ & \quad \left. - \left[\frac{1}{\Psi_{\mathbf{p}^{(0)}}^{(0)} \cdots \Psi_{\mathbf{p}^{(\ell-1)}}^{(\ell-1)}} \Delta_{r_{k-\ell+1},s_{k-\ell+1}} \cdots \Delta_{r_k,s_k} \Phi_{\nu}^{(j)} \right]_{\nu=\lambda^{(k);\gamma_1,\dots,\gamma_{k-\ell-1},1}} \right\} \\ &= \left[\frac{1}{\Psi_{\mathbf{p}^{(0)}}^{(0)} \cdots \Psi_{\mathbf{p}^{(\ell)}}^{(\ell)}} \Delta_{r_{k-\ell},s_{k-\ell}} \cdots \Delta_{r_k,s_k} \Phi_{\nu}^{(j)} \right]_{\nu=\lambda^{(k);\gamma_1,\dots,\gamma_{k-\ell}}} . \end{aligned}$$

This finishes the verification of (2.6.3) by induction. Setting $\ell = j = k$ in (2.6.3) yields

$$\Psi_{\mathbf{p}^{(k)}}^{(k)} = \left[\frac{1}{\Psi_{\mathbf{p}^{(0)}}^{(0)} \cdots \Psi_{\mathbf{p}^{(k-1)}}^{(k-1)}} \Delta_{r_1,s_1} \cdots \Delta_{r_k,s_k} \Phi_{\nu}^{(k)} \right]_{\nu=\lambda^{(k)}}$$

for any $\lambda^{(k)}$ of $n+1$ components and total degree $\delta - k$. We rewrite it as

$$(2.6.4) \quad \Delta_{r_1,s_1} \cdots \Delta_{r_k,s_k} \Phi_{\nu}^{(k)} = \Psi_{\lambda^{(1)},\mathbf{p}^{(1)}}^{(1)} \Psi_{\lambda^{(2)},\mathbf{p}^{(2)}}^{(2)} \cdots \Psi_{\lambda^{(k)},\mathbf{p}^{(k)}}^{(k)}$$

when $\Phi_\nu^{(k)}$ is regarded as a polynomial in the $n + 1$ components of ν . Note that by (2.6.1) the left-hand side

$$\Delta_{r_1, s_1} \cdots \Delta_{r_k, s_k} \Phi_\nu^{(k)}$$

is independent of the value of ν because the degree of $\Phi_\nu^{(k)}$ as a polynomial in ν is no more than k .

Since by Lemma A as a polynomial in ν_1, \dots, ν_n the degree of

$$\Phi_\nu^{(k)} - \left(\sum_{\ell=0}^n \nu_\ell \xi_\ell^{(1)} \right)^k$$

is at most $k - 1$, it follows from (2.6.1) that

$$\Delta_{r_1, s_1} \cdots \Delta_{r_k, s_k} \left(\Phi_\nu^{(k)} - \left(\sum_{\ell=0}^n \nu_\ell \xi_\ell^{(1)} \right)^k \right) = 0$$

and

$$\Delta_{r_1, s_1} \cdots \Delta_{r_k, s_k} \left(\sum_{\ell=0}^n \nu_\ell \xi_\ell^{(1)} \right)^k = \Psi_{\lambda^{(1)}, \mathbf{p}^{(1)}}^{(1)} \Psi_{\lambda^{(2)}, \mathbf{p}^{(2)}}^{(2)} \cdots \Psi_{\lambda^{(k)}, \mathbf{p}^{(k)}}^{(k)}.$$

We have

$$\begin{aligned} & \Delta_{r_k, s_k} \left(\sum_{\ell=0}^n \nu_\ell \xi_\ell^{(1)} \right)^k \\ &= \left(\sum_{\ell=0}^n \nu_\ell \xi_\ell^{(1)} + \xi_{r_k}^{(1)} \right)^k - \left(\sum_{\ell=0}^n \nu_\ell \xi_\ell^{(1)} + \xi_{s_k}^{(1)} \right)^k \\ &= (\xi_{r_k}^{(1)} - \xi_{s_k}^{(1)}) \left\{ \sum_{j=0}^{k-1} \left(\sum_{\ell=0}^n \nu_\ell \xi_\ell^{(1)} + \xi_{r_k}^{(1)} \right)^j \left(\sum_{\ell=0}^n \nu_\ell \xi_\ell^{(1)} + \xi_{s_k}^{(1)} \right)^{k-1-j} \right\}. \end{aligned}$$

Now for $j \geq 1$,

$$\begin{aligned} & \left(\sum_{\ell=0}^n \nu_\ell \xi_\ell^{(1)} + \xi_{r_k}^{(1)} \right)^j \left(\sum_{\ell=0}^n \nu_\ell \xi_\ell^{(1)} + \xi_{s_k}^{(1)} \right)^{k-1-j} - \left(\sum_{\ell=0}^n \nu_\ell \xi_\ell^{(1)} + \xi_{s_k}^{(1)} \right)^{k-1} = \\ & \left(\sum_{\ell=0}^n \nu_\ell \xi_\ell^{(1)} + \xi_{s_k}^{(1)} \right)^{k-1-j} \left\{ \left(\sum_{\ell=0}^n \nu_\ell \xi_\ell^{(1)} + \xi_{r_k}^{(1)} \right)^j - \left(\sum_{\ell=0}^n \nu_\ell \xi_\ell^{(1)} + \xi_{s_k}^{(1)} \right)^j \right\} \\ &= \left(\sum_{\ell=0}^n \nu_\ell \xi_\ell^{(1)} + \xi_{s_k}^{(1)} \right)^{k-1-j} (\xi_{r_k}^{(1)} - \xi_{s_k}^{(1)}) \times \end{aligned}$$

$$\times \left\{ \sum_{\ell=0}^{j-1} \left(\sum_{\ell=0}^n \nu_{\ell} \xi_{\ell}^{(1)} + \xi_{r_k}^{(1)} \right)^{\ell} \left(\sum_{\ell=0}^n \nu_{\ell} \xi_{\ell}^{(1)} + \xi_{s_k}^{(1)} \right)^{j-1-\ell} \right\}$$

is a polynomial of degree no more than $k-2$ in the $n+1$ components of ν . Since

$$\begin{aligned} & \left(\sum_{\ell=0}^n \nu_{\ell} \xi_{\ell}^{(1)} + \xi_{s_k}^{(1)} \right)^{k-1} - \left(\sum_{\ell=0}^n \nu_{\ell} \xi_{\ell}^{(1)} \right)^{k-1} \\ &= \xi_{s_k}^{(1)} \left\{ \sum_{\ell=0}^{k-2} \left(\sum_{\ell=0}^n \nu_{\ell} \xi_{\ell}^{(1)} + \xi_{s_k}^{(1)} \right)^{\ell} \left(\sum_{\ell=0}^n \nu_{\ell} \xi_{\ell}^{(1)} \right)^{k-2-\ell} \right\} \end{aligned}$$

is a polynomial of degree no more than $k-2$ in the $n+1$ components of ν , it follows that

$$\Delta_{r_k, s_k} \left(\sum_{\ell=0}^n \nu_{\ell} \xi_{\ell}^{(1)} \right)^k - k \left(\sum_{\ell=0}^n \nu_{\ell} \xi_{\ell}^{(1)} \right)^{k-1}$$

a polynomial of degree no more than $k-2$ in the $n+1$ components of ν and

$$\Delta_{r_1, s_1} \cdots \Delta_{r_{k-1}, s_{k-1}} \left\{ \Delta_{r_k, s_k} \left(\sum_{\ell=0}^n \nu_{\ell} \xi_{\ell}^{(1)} \right)^k - k \left(\sum_{\ell=0}^n \nu_{\ell} \xi_{\ell}^{(1)} \right)^{k-1} \right\}$$

is zero. By induction on k , we conclude that

$$\Delta_{r_1, s_1} \cdots \Delta_{r_k, s_k} \left(\sum_{\ell=0}^n \nu_{\ell} \xi_{\ell}^{(1)} \right)^k = k! \prod_{\ell=1}^k (\xi_{r_{\ell}}^{(1)} - \xi_{s_{\ell}}^{(1)}).$$

and

$$\Delta_{r_1, s_1} \cdots \Delta_{r_k, s_k} \Phi_{\nu}^{(k)} = k! \prod_{\ell=1}^k (\xi_{r_{\ell}}^{(1)} - \xi_{s_{\ell}}^{(1)}).$$

It follows from (2.6.4) that

$$(2.6.5) \quad \Psi_{\lambda^{(1)}, \mathbf{p}^{(1)}}^{(1)} \Psi_{\lambda^{(2)}, \mathbf{p}^{(2)}}^{(2)} \cdots \Psi_{\lambda^{(k)}, \mathbf{p}^{(k)}}^{(k)} = k! \prod_{\ell=1}^k (\xi_{r_{\ell}}^{(1)} - \xi_{s_{\ell}}^{(1)}).$$

Finally (2.6.5) yields

$$\Psi_{\lambda^{(k)}, \mathbf{p}^{(k)}}^{(k)} = k (\xi_{p_0}^{(1)} - \xi_{p_1}^{(1)})$$

by induction on k for any multi-index $\lambda^{(k)}$ of $n+1$ components with total degree $\delta - k$ and for any binary tree $\mathbf{p}^{(k)}$ of order k which has level-wise homogeneous branching. \square

Remark 2.7. The reason for explicitly computing the function $\Psi_{\lambda^{(k)}, \mathbf{p}^{(k)}}^{(k)}$ is to determine the pole set of the vector field $\Theta_{\lambda^{(k)}, \mathbf{p}^{(k)}}^{(k)}$.

Lemma 2.8. (a) *The function*

$$\Psi_{\lambda^{(k)}, \mathbf{p}^{(k)}}^{(j)} \prod_{\ell=1}^{k-1} (\xi_{r_\ell}^{(1)} - \xi_{s_\ell}^{(1)})$$

is of homogeneous weight j in $\xi_t^{(\ell)}$ ($1 \leq \ell \leq j$, $0 \leq t \leq n$) and is independent of (z_0, \dots, z_n) and is a polynomial in the $n+1$ components of $\lambda^{(k)}$ of degree $\leq j$.

(b) *The vector field*

$$\Theta_{\lambda^{(k)}, \mathbf{p}^{(k)}}^{(j)} \left(\prod_{\ell=1}^k (z_{r_\ell} z_{s_\ell}) \right) \prod_{\ell=1}^{k-1} (\xi_{r_\ell}^{(1)} - \xi_{s_\ell}^{(1)})$$

is a polynomial in z_0, \dots, z_n of degree $\leq k$ and is independent of $\xi_t^{(\ell)}$ ($1 \leq \ell \leq n-1$, $0 \leq t \leq n$). Moreover, the dependence of $\Theta_{\lambda^{(k)}, \mathbf{p}^{(k)}}^{(j)}$ on $\lambda^{(k)}$ is only through the partial differentiation with respect to α_μ with μ depending on $\lambda^{(k)}$.

(c) *The vector field $\Theta_{\lambda^{(k)}, \mathbf{p}^{(k)}}^{(k)}$ is equal to*

$$\frac{z^{\lambda^{(k)}}}{\prod_{\ell=1}^{k-1} (\xi_{r_\ell}^{(1)} - \xi_{s_\ell}^{(1)})} \left[\Delta_{r_1, s_1} \cdots \Delta_{r_k, s_k} \left(z^{-\nu} \frac{\partial}{\partial \alpha_\nu} \right) \right]_{\nu=\lambda^{(k)}},$$

where $\Delta_{r_1, s_1} \cdots \Delta_{r_k, s_k}$ is applied to

$$z^{-\nu} \frac{\partial}{\partial \alpha_\nu}$$

as a function of ν .

2.9. Generation of Vector Fields in the Parameter Direction.

We now look at the special case of Lemma 2.8(c) with $k = n$, then

$$\tilde{\Theta}_{\lambda^{(n)}, \mathbf{p}^{(n)}}^{(n)} = z^{\lambda^{(n)} + \sum_{\ell=1}^n (e_{r_\ell} + e_{s_\ell})} \left[\Delta_{r_1, s_1} \cdots \Delta_{r_n, s_n} \left(z^{-\nu} \frac{\partial}{\partial \alpha_\nu} \right) \right]_{\nu=\lambda^{(n)}}$$

satisfies

$$\tilde{\Theta}_{\lambda^{(n)}, \mathbf{p}^{(n)}}^{(n)} (d^j f) = 0$$

for $0 \leq j \leq n-1$. We fix a point y in X_α , where $\alpha = \{\alpha_\nu\}_{|\nu|=\delta}$. We can choose homogeneous coordinates in \mathbb{P}_n so that

$$(z_0, z_1, \dots, z_n)(y) = (1, 0, \dots, 0).$$

We choose also $s_1 = \dots = s_n = 0$ and $r_j \neq 0$ for $1 \leq j \leq n$. Then at y we end up with

$$\tilde{\Theta}_{\lambda^{(n)}, \mathbf{p}^{(n)}}^{(n)} = \frac{\partial}{\partial \alpha_{\lambda^{(n)} + \sum_{\ell=1}^n e_{r_\ell}}}.$$

For the choice of $\lambda^{(n)}$ we can choose any multi-index of total degree $\delta - n$. When we worry about the generation of the vector fields by global sections, for differentiations in the direction of the parameters $\alpha = \{\alpha_\nu\}_{|\nu|=\delta}$ at the origin, we can capture in inhomogeneous coordinates the differentiation with respect to all coefficients for monomials of degree at least n , because we must include $\sum_{\ell=1}^n e_{r_\ell}$ with $r_j \neq 0$ for $1 \leq j \leq n$ in ν which is equal to $\lambda^{(n)} + \sum_{\ell=1}^n e_{r_\ell}$.

2.10. Example of Vector Fields on Jet Spaces of Low Order.

Let $f = \sum_\nu \alpha_\nu z^\nu$. We introduce

$$\begin{aligned} \xi_j &= dz_j, \\ \xi_j^{(1)} &= \frac{dz_j}{z_j} = \frac{\xi_j}{z_j}. \end{aligned}$$

Then

$$df = \sum_\nu \alpha_\nu \left(\sum_j \nu_j \xi_j^{(1)} \right) z^\nu.$$

Proposition 2.11. *Let $0 \leq p \neq q \leq n$ and $0 \leq r \neq s \leq n$. Let μ be a multi-index of total weight $\delta - 2$. Then*

$$\begin{aligned} & \left\{ \frac{1}{z_r} \left(\left(\frac{1}{z_p} \frac{\partial}{\partial \alpha_{\mu+e_r+e_p}} - \frac{1}{z_q} \frac{\partial}{\partial \alpha_{\mu+e_r+e_q}} \right) \right) \right. \\ & \left. - \frac{1}{z_r} \left(\left(\frac{1}{z_p} \frac{\partial}{\partial \alpha_{\mu+e_r+e_p}} - \frac{1}{z_q} \frac{\partial}{\partial \alpha_{\mu+e_r+e_q}} \right) \right) \right\} (d^j f) = 0 \end{aligned}$$

for $j = 0, 1$.

Proof. We have

$$\left(\frac{1}{z_p} \frac{\partial}{\partial \alpha_{\lambda+e_p}} - \frac{1}{z_q} \frac{\partial}{\partial \alpha_{\lambda+e_q}} \right) (df) = (\xi_q^{(1)} - \xi_p^{(1)}) z^\lambda$$

for any λ with $|\lambda| = \delta - 1$. Choose μ with $|\mu| = \delta - 2$. Apply the above equation to $\lambda = \mu + e_r$ and get

$$\left(\frac{1}{z_p} \frac{\partial}{\partial \alpha_{\mu+e_r+e_p}} - \frac{1}{z_q} \frac{\partial}{\partial \alpha_{\mu+e_r+e_q}} \right) (df) = (\xi_q^{(1)} - \xi_p^{(1)}) z^{\mu+e_r}$$

and

$$\frac{1}{z_r} \left(\left(\frac{1}{z_p} \frac{\partial}{\partial \alpha_{\mu+e_r+e_p}} - \frac{1}{z_q} \frac{\partial}{\partial \alpha_{\mu+e_r+e_q}} \right) \right) (df) = (\xi_q^{(1)} - \xi_p^{(1)}) z^\mu.$$

Since the right-hand side is independent of r , we can replace r by s and take the difference to get

$$\left\{ \frac{1}{z_r} \left(\left(\frac{1}{z_p} \frac{\partial}{\partial \alpha_{\mu+e_r+e_p}} - \frac{1}{z_q} \frac{\partial}{\partial \alpha_{\mu+e_r+e_q}} \right) \right) - \frac{1}{z_r} \left(\left(\frac{1}{z_p} \frac{\partial}{\partial \alpha_{\mu+e_r+e_p}} - \frac{1}{z_q} \frac{\partial}{\partial \alpha_{\mu+e_r+e_q}} \right) \right) \right\} (df) = 0.$$

□

Remark 2.12. We can rewrite the vector field

$$\left\{ \frac{1}{z_r} \left(\left(\frac{1}{z_p} \frac{\partial}{\partial \alpha_{\mu+e_r+e_p}} - \frac{1}{z_q} \frac{\partial}{\partial \alpha_{\mu+e_r+e_q}} \right) \right) - \frac{1}{z_r} \left(\left(\frac{1}{z_p} \frac{\partial}{\partial \alpha_{\mu+e_r+e_p}} - \frac{1}{z_q} \frac{\partial}{\partial \alpha_{\mu+e_r+e_q}} \right) \right) \right\}$$

as

$$z^\mu \left[\Delta_{r,s} \Delta_{p,q} \left(\frac{1}{z^\nu} \frac{\partial}{\partial \alpha_\nu} \right) \right]_{\nu=\mu},$$

where

$$\Delta_{r,s} F(\nu) = F(\nu + e_r) - F(\nu + e_s).$$

To illustrate the situation of vector fields on jet spaces of low order, we do the case of the next order.

$$\begin{aligned} d^2 f &= \sum_{\nu} \alpha_{\nu} \left[\left(\sum_j \nu_j \xi_j^{(1)} \right)^2 + \sum_{\nu} \alpha_j^{(2)} \right] z^{\nu}, \\ \frac{1}{z^{\nu}} \frac{\partial}{\partial \alpha_{\nu}} d^2 f &= \left(\sum_j \nu_j \xi_j^{(1)} \right)^2 + \sum_{\nu} \xi_j^{(2)}, \\ \Delta_{p,q} \left(\frac{1}{z^{\nu}} \frac{\partial}{\partial \alpha_{\nu}} d^2 f \right) &= \left(\sum_j \nu_j \xi_j^{(1)} + \xi_p^{(1)} \right)^2 \\ &\quad - \left(\sum_j \nu_j \xi_j^{(1)} + \xi_q^{(1)} \right)^2 + (\xi_p^{(2)} - \xi_q^{(2)}) \\ &= (\xi_p^{(1)} - \xi_q^{(1)}) \left(2 \sum_j \nu_j \xi_j^{(1)} + \xi_p^{(1)} + \xi_q^{(1)} \right) + (\xi_p^{(2)} - \xi_q^{(2)}), \end{aligned}$$

$$\Delta_{r,s}\Delta_{p,q}\left(\frac{1}{z^\nu}\frac{\partial}{\partial\alpha_\nu}d^2f\right)=2\left(\xi_r^{(1)}-\xi_s^{(1)}\right)\left(\xi_p^{(1)}-\xi_q^{(1)}\right),$$

$$\Delta_{u,v}\Delta_{r,s}\Delta_{p,q}\left(\frac{1}{z^\nu}\frac{\partial}{\partial\alpha_\nu}d^2f\right)=0.$$

Thus for any multi-index μ of total degree $\delta - 3$, the vector field

$$\left[\Delta_{u,v}\Delta_{r,s}\Delta_{p,q}\left(\frac{1}{z^\nu}\frac{\partial}{\partial\alpha_\nu}\right)\right]_{\nu=\mu}$$

annihilates $d^j f$ for $j = 0, 1, 2$, because

$$\left[\frac{\partial}{\partial\alpha_\nu}(d^j f)\right]_{\nu=\lambda}=\left[\frac{\partial}{\partial\alpha_\nu}\right]_{\nu=\lambda}(d^j f).$$

We can now formulate the case of higher-order jets.

Proposition 2.13. *Let $0 \leq r_\ell \neq s_\ell \leq n$ for $1 \leq \ell \leq k$. Let μ be a multi-index of total weight $\delta - k$. Let $\Theta_{\mu;r_1,\dots,r_k;s_1,\dots,s_k}$ denote the vector field*

$$z^\mu \left[\Delta_{r_1,s_1} \cdots \Delta_{r_k,s_k} \left(\frac{1}{z^\nu} \frac{\partial}{\partial\alpha_\nu} \right) \right]_{\nu=\mu}.$$

Then $\Theta_{\mu;r_1,\dots,r_k;s_1,\dots,s_k}(d^j f) = 0$ for $0 \leq j \leq k - 1$.

In the above Proposition the vector field $\Theta_{\mu;r_1,\dots,r_k;s_1,\dots,s_k}$ is a linear combination of the partial differentiation operators

$$\frac{\partial}{\partial\alpha_{\mu+e_{r_{i_1}}+\dots+e_{r_{i_p}}+e_{s_{j_1}}+\dots+e_{s_{j_{k-p}}}}$$

for $0 \leq j \leq k$. The process of generating such vector fields is not independent of coordinate transformations from the general linear group $GL(n+1, \mathbb{C})$. Suppose we have the coordinate transformation

$$z_j = \sum_{\ell=0}^n a_{j\ell} w_\ell \quad (0 \leq j \leq \ell)$$

from the element $\mathbf{a} = (a_{j\ell})_{0 \leq j, \ell \leq n}$ of the general linear group $GL(n+1, \mathbb{C})$. Then

$$z^\nu = z_0^{\nu_0} \cdots z_n^{\nu_n} = \left(\sum_{\ell=0}^n a_{0\ell} w_\ell \right)^{\nu_0} \cdots \left(\sum_{\ell=0}^n a_{n\ell} w_\ell \right)^{\nu_n} = \sum_{|\mu|=\delta} A_{\nu,\mu} w^\mu.$$

Let $(B_{\mu,\nu})_{|\mu|=|\nu|=\delta}$ be the inverse matrix of the matrix $(A_{\nu,\mu})_{|\mu|=|\nu|=\delta}$. Write

$$f = \sum_{|\nu|=\delta} \alpha_\nu z^\nu = \sum_{|\mu|=\delta} \beta_\mu w^\mu.$$

Then

$$\begin{aligned}\beta_\mu &= \sum_{|\nu|=\delta} \alpha_\nu A_{\nu,\mu}, \\ \alpha_\mu &= \sum_{|\mu|=\delta} \beta_\mu B_{\mu,\nu}.\end{aligned}$$

When the generation of the vector field Θ in the coordinate system (z_0, \dots, z_n) gives

$$\Theta_z = \sum_\nu g_\nu(z) \frac{\partial}{\partial \alpha_\nu},$$

the procedure applied to the coordinate system (w_0, \dots, w_n) gives

$$\Theta_w = \sum_\mu g_\mu(w) \frac{\partial}{\partial \beta_\mu}.$$

When we transform back to the coordinate system (z_0, \dots, z_n) , we get

$$\Theta_{\mathbf{a}; \mu; r_1, \dots, r_n; s_1, \dots, s_n} = \sum_{\mu, \nu} g_\mu(w(z)) \frac{\partial \alpha_\nu}{\partial \beta_\mu} \frac{\partial}{\partial \alpha_\nu}.$$

Now We would like to show that when $k = n$, the dimension of the quotient space

$$\left(\bigoplus_{|\nu|=\delta} \mathbb{C} \frac{\partial}{\partial \alpha_\nu} \right) / \left(\sum_{\substack{\mathbf{a} \in GL(n+1, \mathbb{C}); |\mu|=\delta-n; \\ r_1, \dots, r_n; s_1, \dots, s_n}} \mathbb{C} \Theta_{\mathbf{a}; \mu; r_1, \dots, r_n; s_1, \dots, s_n} \right)$$

is no more than n over \mathbb{C} . For this we need only show that modulo the linear space generated by all such $\Theta_{\mathbf{a}; \mu; r_1, \dots, r_n; s_1, \dots, s_n}$, every generator $\frac{\partial}{\partial \alpha_\nu}$ can be expressed as a linear combination of n fixed

$$\frac{\partial}{\partial \alpha_{\nu^{[\ell]}}} \quad (1 \leq \ell \leq n),$$

where $\nu^{[\ell]}$ is a multi-index of total weight δ . We use the linear transformations defined by $\mathbf{a} \in GL(n+1, \mathbb{C})$ simply to make sure that, for any given point, we are free to do the checking in an appropriate coordinate system which depends on the point j .

For the convenience of bookkeeping we let M be an integer $> \delta$ and introduce a new weight $\|\nu\|_M$ for any multi-index ν of total degree δ which is defined as follows.

$$\|\nu\|_M = \sum_{\ell=0}^n \nu_\ell M^\ell.$$

We single out the n multi-index ν of total degree δ which has the n lowest weight $\|\nu\|_M$ possible, namely,

$$\delta - \ell + (n - 1 - \ell)M \quad (0 \leq \ell \leq n - 1).$$

These $n - 1$ multi-indices are

$$\nu^{[\ell]} = (\delta - \ell, n - 1 - \ell, 0, \dots, 0) \quad (0 \leq \ell \leq n - 1).$$

Fix a point P_0 in the space $J_{n-1}^{\text{vert}}(\mathcal{X})$ of vertical $(n - 1)$ -jets. Choose a coordinate system (z_0, \dots, z_n) so that all the coefficients of

$$\frac{\partial}{\partial \alpha_{\mu + e_{r_{i_1}} + \dots + e_{r_{i_p}} + e_{s_{j_1}} + \dots + e_{s_{j_{k-p}}}}$$

occurring in

$$\Theta_{\mu; r_1, \dots, r_n; s_1, \dots, s_n} = z^\mu \left[\Delta_{r_1, s_1} \cdots \Delta_{r_k, s_k} \left(\frac{1}{z^\nu} \frac{\partial}{\partial \alpha_\nu} \right) \right]_{\nu=\mu}$$

are all nonzero. Then modulo $\Theta_{\mu; r_1, \dots, r_n; s_1, \dots, s_n}$ we can express

$$\frac{\partial}{\partial \alpha_{\mu + e_{r_1} + \dots + e_{r_n}}}$$

as a linear combination of

$$\frac{\partial}{\partial \alpha_{\mu + e_{r_{i_1}} + \dots + e_{r_{i_p}} + e_{s_{j_1}} + \dots + e_{s_{j_{k-p}}}}$$

for $p < n$. Now take any multi-index ν with total degree δ which is different from any one of $\nu^{[0]}, \dots, \nu^{[n-1]}$. In other words,

$$\|\nu\|_M > \delta - n + 1 + (n - 1)M.$$

Then for some $1 \leq r_1, \dots, r_n \leq n$, all the $n + 1$ components of

$$\nu - \sum_{\ell=1}^n e_{r_\ell}$$

are nonnegative. Let

$$\begin{aligned} \mu &= \nu - \sum_{\ell=1}^n e_{r_\ell}, \\ s_1 &= \dots = s_n = 0. \end{aligned}$$

Then modulo $\Theta_{\mu; r_1, \dots, r_n; s_1, \dots, s_n}$ we can express

$$\frac{\partial}{\partial \alpha_\nu}$$

in terms of

$$\frac{\partial}{\partial \alpha_{\mu + e_{r_{i_1}} + \dots + e_{r_{i_p}} + e_{s_{j_1}} + \dots + e_{s_{j_{k-p}}}}$$

for $p < n$ with

$$\left\| \mu + e_{r_{i_1}} + \dots + e_{r_{i_p}} + e_{s_{j_1}} + \dots + e_{s_{j_{k-p}}} \right\|_M < \|\nu\|_M.$$

We thus conclude that modulo

$$\sum_{\substack{\mathbf{a} \in GL(n+1, \mathbb{C}); |\mu| = \delta - n; \\ r_1, \dots, r_n; s_1, \dots, s_n}} \mathbb{C} \Theta_{\mathbf{a}; \mu; r_1, \dots, r_n; s_1, \dots, s_n},$$

the space

$$\bigoplus_{|\nu| = \delta} \mathbb{C} \frac{\partial}{\partial \alpha_\nu}$$

is generated by

$$\frac{\partial}{\partial \alpha_{\nu[0]}}, \dots, \frac{\partial}{\partial \alpha_{\nu[n-1]}}$$

and we conclude that the dimension of the quotient space

$$\left(\bigoplus_{|\nu| = \delta} \mathbb{C} \frac{\partial}{\partial \alpha_\nu} \right) / \left(\sum_{\substack{\mathbf{a} \in GL(n+1, \mathbb{C}); |\mu| = \delta - n; \\ r_1, \dots, r_n; s_1, \dots, s_n}} \mathbb{C} \Theta_{\mathbf{a}; \mu; r_1, \dots, r_n; s_1, \dots, s_n} \right)$$

is no more than n over \mathbb{C} . Note that the pole order of each of the meromorphic vector field

$$\Theta_{\mathbf{a}; \mu; r_1, \dots, r_n; s_1, \dots, s_n}$$

is no more than $2n$ along the infinity hyperplane of \mathbb{P}_n .

Remark 2.14. The reason why in the above argument we fail to get generation of all vectors in parameter space is that we can only expect to get generation all vectors in parameter space up to codimension n for $(n-1)$ -jets. The vector fields have to be tangential to the space $J_{n-1}^{(\text{vert})}(\mathcal{X})$ of vertical $(n-1)$ -jets of \mathcal{X} which is of codimension n in the product

$$J_{n-1}(\mathbb{P}_n) \times \mathbb{P}_N$$

of the space $J_{n-1}(\mathbb{P}_n)$ of $(n-1)$ -jets of \mathbb{P}_n and the parameter space \mathbb{P}_N .

2.15. Generation of Vectors in Vertical Directions

Now we construct holomorphic vector fields which generate the vertical directions modulo the horizontal directions.

$$z_j \frac{\partial}{\partial z_j} (d^k f) = \sum_{\nu} \alpha_{\nu} \nu_j \Phi_{\nu}^{(k)} (\xi^{(1)}, \dots, \xi^{(k)}) z^{\nu},$$

$$\frac{\partial}{\partial \xi_j^{\ell}} (d^k f) = \sum_{\nu} \alpha_{\nu} \left[\frac{\partial}{\partial \xi_j^{\ell}} \Phi_{\nu}^{(k)} (\xi^{(1)}, \dots, \xi^{(k)}) \right] z^{\nu}.$$

To unify the notations, we use T to denote any one of

$$z_j \frac{\partial}{\partial z_j}, \quad \frac{\partial}{\partial \xi_j^{\ell}} \quad (0 \leq j \leq n, 1 \leq \ell \leq n)$$

and write

$$T (d^k f) = - \sum_{\nu} \Xi_{\nu}^{(k)} z^{\nu}.$$

This means that

$$\Xi_{\nu}^{(k)} = -\alpha_{\nu} \nu_j \Phi_{\nu}^{(k)} (\xi^{(1)}, \dots, \xi^{(k)}) \quad \text{when } T = z_j \frac{\partial}{\partial z_j}$$

and

$$\Xi_{\nu}^{(k)} = -\alpha_{\nu} \left[\frac{\partial}{\partial \xi_j^{\ell}} \Phi_{\nu}^{(k)} (\xi^{(1)}, \dots, \xi^{(k)}) \right] \quad \text{when } T = \frac{\partial}{\partial \xi_j^{\ell}}.$$

The function $\Xi_{\nu}^{(k)}$ is homogeneous of weight $k - \ell$ and is independent of z_0, \dots, z_n .

We also unify the notations for the vector fields $\Theta_{\lambda^{(k)}, \mathbf{p}^{(k)}}^{(k)}$ and write $\Theta_{\nu}^{(k)}$ as the vector field with effective low pole order such that

$$\begin{aligned} \Theta_{\nu}^{(k)} (d^j f) &= 0 \text{ for } 0 \leq j < k, \\ \Theta_{\nu}^{(k)} (d^k f) &= z^{\nu}, \\ \Theta_{\nu}^{(k)} (d^j f) &= -\Psi_{\nu}^{(k,j)} z^{\nu} \text{ for } j > k. \end{aligned}$$

This means the following. Choose a binary tree $\mathbf{p}^{(n-1)}$ of order $n - 1$ which has level-wise homogeneous branches. Let $r_j = p_{\gamma_1, \dots, \gamma_{j-1}, 0}$ and $s_j = p_{\gamma_1, \dots, \gamma_{j-1}, 1}$ for $1 \leq j \leq n - 1$. Since the binary tree $\mathbf{p}^{(n-1)}$ of order $n - 1$ has level-wise homogeneous branches, the values of $r_j = p_{\gamma_1, \dots, \gamma_{j-1}, 0}$ and $s_j = p_{\gamma_1, \dots, \gamma_{j-1}, 1}$ are independent of the choices of the values 0 or 1 for $\gamma_1, \dots, \gamma_{j-1}$ and $1 \leq j \leq n - 1$. Let $\mathbf{p}^{(n-1-j)} = \mathbf{p}^{(n-1; \gamma_1, \dots, \gamma_j)}$ for $1 \leq j \leq n - 1$. We know that $\mathbf{p}^{(n-1-j)}$ is independent of the choices of

the values 0 or 1 for $\gamma_1, \dots, \gamma_j$ and $0 \leq j \leq n$ because the binary tree $\mathbf{p}^{(n-1)}$ of order $n-1$ has level-wise homogeneous branches.

Given any multi-index ν of $n+1$ components and total degree δ , we choose a multi-index $\lambda_\nu^{(n-1)}$ of $n+1$ components and total degree $\delta-n+1$ such that $\lambda_\nu^{(n-1)} \leq \nu$ in the sense that the j -th component of $\lambda_\nu^{(n-1)} \leq \nu_j$ is no more than ν_j for $0 \leq j \leq n$. Though the binary tree $\mathbf{p}^{(n-1)}$ of order $n-1$ has level-wise homogeneous branches, yet $\lambda_\nu^{(n-1; \gamma_1, \dots, \gamma_j)}$ does depend on the choices of the values 0 or 1 for $\gamma_1, \dots, \gamma_j$ and $1 \leq j \leq n$. The dependence is as follows. If we denote r_j by $r_{j,0}$ and $s_j = r_{j,1}$, then

$$\lambda_\nu^{(n-1; \gamma_1, \dots, \gamma_j)} = \lambda_\nu^{(n-1-j)} = \lambda_\nu^{(n-1)} + \sum_{\ell=1}^j e_{r_\ell, \gamma_\ell}$$

By Lemma 2.4,

$$\Theta_{\lambda^{(k)}, \mathbf{p}^{(k)}}^{(k)} (d^j f) = z^{\lambda^{(k)}} \Psi_{\lambda^{(k)}, \mathbf{p}^{(k)}}^{(j)} \quad \text{for } k \leq j \leq n$$

for any multi-index $\lambda^{(k)}$ of $n+1$ components and total degree $\delta-k$. So we can set

$$\begin{aligned} \Theta_\nu^{(k)} &= \frac{-z^{\nu-\lambda_\nu^{(n-1-k)}}}{\Psi_{\mathbf{p}^{(k)}}^{(k)}} \Theta_{\lambda_\nu^{(n-1-k; \gamma_1, \dots, \gamma_k)}, \mathbf{p}^{(k)}}^{(k)}, \\ \Psi_\nu^{(k,j)} &= \frac{-z^{\nu-\lambda_\nu^{(n-1-k)}}}{\Psi_{\mathbf{p}^{(k)}}^{(k)}} \Psi_{\lambda_\nu^{(n-1-k; \gamma_1, \dots, \gamma_k)}, \mathbf{p}^{(k)}}^{(j)}. \end{aligned}$$

By Lemma 2.8, The function

$$\Psi_\nu^{(k,j)} \prod_{\ell=1}^k (\xi_{r_\ell}^{(1)} - \xi_{s_\ell}^{(1)})$$

is of homogeneous weight j in $\xi_t^{(\ell)}$ ($1 \leq \ell \leq j$, $0 \leq t \leq n$) and is a polynomial in the variables z_0, \dots, z_n of degree $\leq n-1-k$ and in the $n+1$ components of ν of degree $\leq j$. The vector field

$$\Theta_\nu^{(k)} \prod_{\ell=1}^k (z_{r_\ell} z_{s_\ell} (\xi_{r_\ell}^{(1)} - \xi_{s_\ell}^{(1)}))$$

is a polynomial in z_0, \dots, z_n of degree $\leq n-1$ and is independent of $\xi_t^{(\ell)}$ ($1 \leq \ell \leq n-1$, $0 \leq t \leq n$). Moreover, the dependence of $\Theta_{\lambda^{(k)}, \mathbf{p}^{(k)}}^{(j)}$ on ν is only through the partial differentiation with respect to α_μ with μ depending on ν .

To make sure that a modification of T annihilates f , we modify T to $T + \sum_{\nu} \Xi_{\nu}^{(0)} \Theta_{\nu}^{(0)}$. To make sure that our constructed vector field annihilates df , we use

$$\left(T + \sum_{\nu} \Xi_{\nu}^{(0)} \Theta_{\nu}^{(0)} \right) (df) = - \sum_{\nu} \Xi_{\nu}^{(0)} \Psi_{\nu}^{(0,1)} z^{\nu} - \sum_{\nu} \Xi_{\nu}^{(1)} z^{\nu}.$$

This means that we have to modify $T + \sum_{\nu} \Xi_{\nu}^{(0)} \Theta_{\nu}^{(0)}$ to

$$T + \sum_{\nu} \Xi_{\nu}^{(0)} \Theta_{\nu}^{(0)} + \sum_{\nu} \Xi_{\nu}^{(0)} \Psi_{\nu}^{(0,1)} \Theta_{\nu}^{(1)} + \sum_{\nu} \Xi_{\nu}^{(1)} \Theta_{\nu}^{(1)}.$$

To go one step further to make sure that our constructed vector field annihilates $d^2 f$, we use

$$\begin{aligned} & \left(T + \sum_{\nu} \Xi_{\nu}^{(0)} \Theta_{\nu}^{(0)} + \sum_{\nu} \Xi_{\nu}^{(0)} \Psi_{\nu}^{(0,1)} \Theta_{\nu}^{(1)} + \sum_{\nu} \Xi_{\nu}^{(1)} \Theta_{\nu}^{(1)} \right) (d^2 f) \\ &= - \sum_{\nu} \Xi_{\nu}^{(0)} \Psi_{\nu}^{(0,2)} z^{\nu} - \sum_{\nu} \Xi_{\nu}^{(0)} \Psi_{\nu}^{(0,1)} \Psi_{\nu}^{(1,2)} z^{\nu} - \sum_{\nu} \Xi_{\nu}^{(1)} \Psi_{\nu}^{(1,2)} z^{\nu} - \sum_{\nu} \Xi_{\nu}^{(2)} z^{\nu}. \end{aligned}$$

Thus we have to modify

$$T + \sum_{\nu} \Xi_{\nu}^{(0)} \Theta_{\nu}^{(0)} + \sum_{\nu} \Xi_{\nu}^{(0)} \Psi_{\nu}^{(0,1)} \Theta_{\nu}^{(1)} + \sum_{\nu} \Xi_{\nu}^{(1)} \Theta_{\nu}^{(1)}$$

to

$$\begin{aligned} & T + \sum_{\nu} \Xi_{\nu}^{(0)} \Theta_{\nu}^{(0)} + \sum_{\nu} \Xi_{\nu}^{(0)} \Psi_{\nu}^{(0,1)} \Theta_{\nu}^{(1)} \\ & + \sum_{\nu} \Xi_{\nu}^{(0)} \Psi_{\nu}^{(0,2)} \Theta_{\nu}^{(2)} + \sum_{\nu} \Xi_{\nu}^{(0)} \Psi_{\nu}^{(0,1)} \Psi_{\nu}^{(1,2)} \Theta_{\nu}^{(2)} \\ & + \sum_{\nu} \Xi_{\nu}^{(1)} \Theta_{\nu}^{(1)} + \sum_{\nu} \Xi_{\nu}^{(1)} \Psi_{\nu}^{(1,2)} \Theta_{\nu}^{(2)} + \sum_{\nu} \Xi_{\nu}^{(2)} \Theta_{\nu}^{(2)}. \end{aligned}$$

In general, to make sure that we have the annihilation of all $d^j f$ for $0 \leq j \leq n-1$, we need to write

$$T + \sum_{\nu} \sum_{0 \leq j_0 \leq k \leq n-1} \Xi_{\nu}^{(j_0)} \left(\sum_{\ell=0}^{k-j_0-1} \sum_{j_0 < \dots < j_{\ell} < k} \Psi_{\nu}^{(j_{\ell}, k)} \prod_{q=0}^{\ell-1} \Psi_{\nu}^{(j_q, j_{q+1})} \right) \Theta_{\nu}^{(k)}.$$

The main point is to control the pole order of the vector fields and make the pole order bounded and independent of δ . That is the reason why we want to remove z^{ν} by using the vector fields $\Theta_{\lambda^{(k)}, \mathbf{p}^{(k)}}^{(k)}$. We now count the degree in z_0, \dots, z_n and the weight in $\xi_j^{(\ell)}$ after we clear the

denominators. We need to multiply

$$\Psi_{\nu}^{(j_{\ell}, k)} \left(\prod_{q=0}^{\ell-1} \Psi_{\nu}^{(j_q, j_{q+1})} \right) \Theta_{\nu}^{(k)}$$

by

$$\left[\prod_{q=0}^{\ell} \left(\prod_{i=1}^{j_q} (\xi_{r_i}^{(1)} - \xi_{s_i}^{(1)}) \right) \right] \prod_{i=1}^k (\xi_{r_i}^{(1)} - \xi_{s_i}^{(1)})$$

to get rid of the denominator involving $\xi_j^{(\ell)}$. The worst that can occur is

$$\prod_{\ell=1}^k (\xi_{r_{\ell}}^{(1)} - \xi_{s_{\ell}}^{(1)})^{n-\ell},$$

whose weight is $\frac{n(n-1)}{2}$. Since

$$\Psi_{\nu}^{(k, j)} \prod_{\ell=1}^k (\xi_{r_{\ell}}^{(1)} - \xi_{s_{\ell}}^{(1)})$$

is of homogeneous weight j in $\xi_t^{(\ell)}$ ($1 \leq \ell \leq j$, $0 \leq t \leq n$), it follows that the worst situation is that after multiplication by the above factor to clear the denominator we end up with a weight of $j_0 + j_1 + \cdots + j_{\ell}$ plus the weight of the factor which is no greater than the weight

$$j_0 + j_1 + \cdots + j_{\ell} + \frac{n(n-1)}{2} \leq n(n-1).$$

When it comes to the degree in z_0, \dots, z_n , we have degree 1 from T , multiplication by the factor

$$\prod_{\ell=1}^k (z_{r_{\ell}} z_{s_{\ell}})$$

to clear the denominator of $\Theta_{\nu}^{(k)}$ to yield degree $\leq n-1$, and the degree of

$$\Psi_{\nu}^{(k, j)} \prod_{\ell=1}^k (\xi_{r_{\ell}}^{(1)} - \xi_{s_{\ell}}^{(1)})$$

no more than $n-1-k$. So after clearing the denominators, we have no more than $2(n-1)+1$ in degree for T and no more than

$$\begin{aligned} & (n-1-j_0) + \cdots + (n-1-j_k) + (n-1) + 2n \\ & \leq n(n-1) + (n-1) + 2n \leq n^2 - 1 + 2n \end{aligned}$$

for

$$\Psi_{\nu}^{(j_{\ell}, k)} \left(\prod_{q=0}^{\ell-1} \Psi_{\nu}^{(j_q, j_{q+1})} \right) \Theta_{\nu}^{(k)}.$$

Finally we conclude that, after clearing the denominators, we end up with weight no more than $n(n-1)$ in $\xi_t^{(\ell)}$ ($1 \leq \ell \leq j$, $0 \leq t \leq n$) and degree no more than $n^2 - 1 + 2n$ in z_0, \dots, z_n .

2.16. Vector Fields in Terms of Differentiation with Respect to Inhomogeneous Coordinates.

The introduction of homogeneous coordinates is simply for the notational convenience of our discussion. We now return to inhomogeneous coordinates by specializing to $z_0 \equiv 1$. First of all we would like to go back to the coordinates

$$d^j z_{\ell} \quad (0 \leq j \leq n-1, 0 \leq \ell \leq n)$$

from the coordinates

$$z_0, z_1, \dots, z_n, \\ d^j \log z_{\ell} \quad (1 \leq j \leq n-1, 0 \leq \ell \leq n)$$

which is the same as

$$z_0, z_1, \dots, z_n, \\ \xi_{\ell}^{(j)} \quad (1 \leq j \leq n-1, 0 \leq \ell \leq n)$$

(because $\xi_j^{(k)} = d^k \log z_j$). We are going to use the chain rule for the transformation of vector fields.

$$\frac{\partial}{\partial (d^{\ell} z_p)} = \sum_{j,k} \frac{\partial \xi_j^{(k)}}{\partial (d^{\ell} z_p)} \frac{\partial}{\partial \xi_j^{(k)}}.$$

Since $\xi_j^{(k)} = d^{k-1} \left(\frac{dz_j}{z_j} \right)$, it follows that

$$\xi_j^{(k)} = \xi_j^{(k)} (z_0, \dots, z_n, dz_0, \dots, dz_n, \dots, d^n z_0, \dots, d^n z_n)$$

is a rational function which is homogeneous of weight 0 when $d^{\ell} z_p$ is assigned weight 1 and is homogeneous of weight k when $d^{\ell} z_p$ is assigned weight ℓ . Thus

$$\frac{\partial \xi_j^{(\ell)}}{\partial (d^{\ell} z_p)}$$

is of weight -1 when $d^\ell z_p$ is assigned weight 1 and is homogeneous of weight $k - \ell$ when $d^\ell z_p$ is assigned weight ℓ . It follows from weight considerations that

$$\frac{\partial \xi_j^{(\ell)}}{\partial (d^\ell z_p)} = \delta_{p,j} \left(\frac{1}{z_j} \right),$$

where $\delta_{p,j}$ is the Kronecker delta. We conclude that, so far as the independence of the constructed vector fields are concerned, it makes no difference whether we are using the coordinate system

$$z_0, z_1, \dots, z_n, \\ d^j \log z_\ell \quad (1 \leq j \leq n-1, 0 \leq \ell \leq n)$$

or the coordinate system

$$z_0, z_1, \dots, z_n, \\ \xi_\ell^{(j)} \quad (1 \leq j \leq n-1, 0 \leq \ell \leq n).$$

Now we pass from the homogeneous coordinates to the inhomogeneous coordinates. It is equivalent to restricting all the objects to the linear subspace

$$z_0 = 1, dz_0 = d^2 z_0 = \dots = d^n z_0 = 0.$$

So far as

$$\frac{\partial}{\partial \xi_j^{(\ell)}}, \quad \frac{\partial}{\partial (d^\ell z_p)}$$

are concerned, the linear subspace is part of the line defined by setting some coordinates equal to constant and the argument is not affected. Since the pole order of $d^\ell z_j$ is no more than $\ell + 1$, we have the following proposition. A point of the space $J_{n-1}^{\text{vert}}(\mathcal{X})$ of vertical $(n-1)$ -jets is represented by a nonsingular complex curve germ in \mathbb{P}_n precisely when the value of $z_1 dz_0 - z_0 dz_1$ is nonzero at it for some homogeneous coordinate system z_0, \dots, z_n of \mathbb{P}_n .

Proposition 2.17. (Global Generation on Jet Space by Slanted Vector Fields at Points Representable by Regular Curve Germs) *Let P_0 be a point of the space $J_{n-1}^{\text{vert}}(\mathcal{X})$ of vertical $(n-1)$ -jets such that P_0 can be represented by a nonsingular complex curve germ in \mathbb{P}_n . Then the meromorphic vector fields on $\mathbb{P}_n \times \mathbb{P}_N$ tangential to*

$$\{f = df = \dots = d^n f = 0\}$$

of pole order $\leq n^2 + 2n + n(n-1) = n(2n+1)$ (along the infinity hyperplane of \mathbb{P}_n) generate at P_0 the tangent space of the total space of fiber-direction $(n-1)$ -jets, where $d^k f$ is taken with α_ν regarded as constants. In terms of inhomogeneous coordinates, the statement is

equivalent to that of the formulation in terms of homogeneous coordinates on the restriction to

$$\{z_j = 1, dz_j = d^2 z_j = \cdots = d^n z_j = 0\}$$

for each $0 \leq j \leq n$.

Proof. Since P_0 can be represented by a nonsingular complex curve germ in \mathbb{P}_n , there exists some homogeneous coordinate system z_0, \dots, z_n such that $z_0 \neq 0$ and $z_1 dz_0 - z_0 dz_1 \neq 0$ at it. Since one of z_0 and z_1 must be nonzero at P_0 , we assume without loss of generality that z_0 is nonzero at P_0 . Let $x_j = \frac{z_j}{z_0}$ for $1 \leq j \leq n$. Then $dx_1 = \frac{z_0 dz_1 - z_1 dz_0}{z_0^2} \neq 0$ at P_0 . We then apply a translation to the affine coordinates (x_1, \dots, x_n) to make sure that $x_\ell \neq 0$ at P_0 for $1 \leq \ell \leq n$. Then we apply a linear transformation to the affine coordinates (x_1, \dots, x_n) that $x_\ell \neq 0$ at P_0 for $1 \leq \ell \leq n$ and $dx_1 \neq dx_2$ at P_0 . We are going to set $r_j = 1$ and $s_j = 2$ for $1 \leq j \leq n$. Also we will restrict the vector fields to $z_0 = 1$ and $z_\ell = x_\ell$ for $1 \leq \ell \leq n$ so that

$$\xi_{r_j}^{(1)} - \xi_{s_j}^{(1)} = \xi_1^{(1)} - \xi_2^{(1)} \neq 0$$

at P_0 and $z_\ell \neq 0$ at P_0 for $0 \leq \ell \leq n$. The above construction now gives the generation of the tangent bundle of $J_{n-1}^{\text{vert}}(\mathcal{X})$ at P_0 . \square

Remark 2.18. In the global generation of the tangent bundle of \mathcal{X} in Lemma 1.7 there is no reference to the tangent vector be representable by nonsingular complex curve germ, because a tangent vector is not representable by a nonsingular complex curve germ must be zero and the identically zero global vector field already generates the zero tangent vector. However, a higher order jet which cannot be represented by a nonsingular complex curve germ need not be zero. The condition of representability by a nonsingular complex curve germ can be technically suppressed by formulating global generation over some suitably defined projectivization of the jet space which includes only those jets which have well-defined images in the projectivization of the tangent bundle.

For the hyperbolicity of generic hypersurface X of sufficiently high degree, the generation by slanted vector fields of low vertical pole order only at jets representable by nonsingular complex curve germs offers no difficulty, because any nonconstant holomorphic map φ from the affine complex line \mathbb{C} to X must be nonzero tangent vector at some point ζ_0 of \mathbb{C} and we need use slanted vector fields of low vertical pole order at the jet represented by φ at $\varphi(\zeta_0)$.

The slanted vector fields on $J_{n-1}^{\text{vert}}(\mathcal{X})$ of low vertical pole order constructed in (2.15) for the the proof of Proposition 2.17 start out with the vector field T which is any one of

$$z_j \frac{\partial}{\partial z_j}, \quad \frac{\partial}{\partial \xi_j^\ell} \quad (0 \leq j \leq n, 1 \leq \ell \leq n).$$

The constructed slanted vector fields on $J_{n-1}^{\text{vert}}(\mathcal{X})$ of low vertical pole order are actually restrictions $J_{n-1}^{\text{vert}}(\mathcal{X})$ of vector fields on $J_{n-1}(\mathbb{P}_n) \times \mathbb{P}_N$ of low vertical pole order which are tangential to $J_{n-1}^{\text{vert}}(\mathcal{X})$. We formulate below a proposition about the slanted vector fields on $J_{n-1}(\mathbb{P}_n) \times \mathbb{P}_N$ of low vertical pole order which are tangential to $J_{n-1}^{\text{vert}}(\mathcal{X})$ before their restrictions to $J_{n-1}^{\text{vert}}(\mathcal{X})$. We want generation of the tangent space of $J_{n-1}(\mathbb{P}_n) \times \mathbb{P}_N$ only at points P_0 represented by a nonsingular complex curve germ in $\mathbb{P}_n \times \{\alpha\}$ which is disjoint from \mathcal{X} . In some steps we need to modify the proof of Proposition 2.17 by replacing the defining function f of \mathcal{X} by its product with a polynomial g of degree n in x_1, \dots, x_n in order to make sure that

$$\{fg = d(fg) = \dots = d^n(fg) = 0\}$$

is satisfied at P_0 , where $d^k(fg)$ is taken with α_ν regarded as constants.

Proposition 2.19. *Let $\alpha \in \mathbb{P}_N$ and P_0 be a point of the space $J_{n-1}(\mathbb{P}_n) \times \mathbb{P}_N$ such that P_0 can be represented by a nonsingular complex curve germ in $\mathbb{P}_n \times \{\alpha\}$ which is disjoint from \mathcal{X} . Then the meromorphic vector fields on $J_{n-1}(\mathbb{P}_n) \times \mathbb{P}_N$ of pole order $\leq n^2 + 2n + n(n-1) = n(2n+1)$ (along the infinity hyperplane of \mathbb{P}_n) which are tangential to $J_{n-1}^{\text{vert}}(\mathcal{X})$ generate at P_0 the tangent space of $J_{n-1}(\mathbb{P}_n) \times \mathbb{P}_N$.*

2.20. Remark on Homogeneous Weights of Vector Fields.

We would like to consider the effect on the weight of a jet differential after the application of the vector fields which we have constructed. The weight of $d^j x_\ell$ is j . The coordinates x_ℓ has weight zero and does not contribute at all to the computation of weights. When we consider the vector field which starts with

$$\frac{\partial}{\partial (d^k x_\ell)},$$

to clear the denominator we have to multiply the result by the factor

$$\left(\frac{dx_1}{x_1} - \frac{dx_2}{x_2} \right)^k$$

so that one ends up with

$$\left(\frac{dx_1}{x_1} - \frac{dx_2}{x_2}\right)^k \frac{\partial}{\partial (d^k x_\ell)}$$

which means that the action of the vector field after clearing out the denominator preserves the weight of the jet differential. Moreover, by the explicit construction of the slanted vector fields of low vertical pole order, we cannot apply them to $(n-1)$ -jet differentials to get k -jet differentials for $k < n-1$.

By applying slanted vector fields of low vertical pole order (from Proposition 2.19), at points of X where the $(n-1)$ -jet differentials (assumed to be holomorphically extendible to holomorphic jet differentials on neighboring hypersurfaces $X^{(\alpha)}$) are not zero pointwise, we can lower the multiplicity of the common zero-set of all holomorphic $(n-1)$ -jet differentials vanishing on an ample divisor as functions of $d^\ell x_j$ ($1 \leq j \leq n$, $1 \leq \ell \leq n-1$) with x_1, \dots, x_{n-1} being regarded as local coordinates of X to yield already at this point the conclusion that, for the degree δ of X sufficiently large, there is a proper subvariety of X which contains the image of all nonconstant holomorphic maps from \mathbb{C} to X , as explained in (2.21) below, under the assumption that, for some $0 < \theta < \theta' \leq 1$, there is a holomorphic $(n-1)$ -jet differential on X whose weight m is of order at most comparable to δ^θ and whose order of vanishing at the intersection of X and the infinity hyperplane of \mathbb{P}_n is of order at least comparable to $\delta^{\theta'}$.

2.21. Non Zariski Density of Entire Curves from Existence of Appropriate Jet Differential .

We have vector fields on $J_k(\mathbb{P}_n) \times \mathbb{P}_N$ with low pole-order which are tangential to the space

$$J_{n-1}^{\text{vert}}(\mathcal{X}) = \bigcup_{\alpha \in \mathbb{P}_N} J_k(X^{(\alpha)})$$

of vertical $(n-1)$ -jets of \mathcal{X} . We can use either homogeneous coordinates z_0, \dots, z_n of \mathbb{P}_n or inhomogeneous coordinates x_1, \dots, x_n . The difference is very low pole order and such a very low pole order does not affect our argument. Then we use the affine coordinates $\xi_j^{(\ell)} = d^\ell \log z_j$ or $d^\ell x_j$. A k -jet differential of weight m is simply a polynomial in $\xi_j^{(\ell)}$ homogeneous of weight m , which is pulled back to $z_0 = 1$, $z_j = x_j$ when the affine coordinates are used. Assume that the k -jet differential is holomorphically extendible to holomorphic jet differentials on neighboring hypersurfaces $X^{(\alpha)}$. We can either use Lie derivatives with

respect to the vector fields or we could just use differentiation of functions on the variables z_0, \dots, z_n (or x_1, \dots, x_n) and $\xi_j^{(\ell)}$ (or dx_j). Even after we clear the denominator of the vector field used, with differentiation with respect to the vector field we end up with another polynomial in $\xi_j^{(\ell)}$ homogeneous of another weight m' .

Suppose $0 < \theta < \theta' \leq 1$ and we have a holomorphic $(n-1)$ -jet differential $\omega^{(\hat{\alpha})}$ on $X^{(\hat{\alpha})}$ (assumed to be holomorphically extendible to holomorphic jet differentials on neighboring hypersurfaces $X^{(\alpha)}$) whose weight m is of order at most comparable to δ^θ and whose order of vanishing at the intersection of $X^{(\hat{\alpha})}$ and the infinity hyperplane of \mathbb{P}_n is of order at least comparable to $\delta^{\theta'}$. Let Z be the set of $X^{(\hat{\alpha})}$ where all coefficients of the $(n-1)$ -jet differential ω vanishes.

At a point y of $X - Z$ the $(n-1)$ -jet differential ω of weight m , as a function on the space $J_{n-1}(X)$ of $(n-1)$ -jets of $X^{(\hat{\alpha})}$, has vanishing order at most m at every point of $J_{n-1}(X)$ above y , because at least one of the coefficients of ω is nonzero. Because of $\theta < \theta'$, at any point P_0 , above y , of $J_{n-1}(X)$ which is represented by a nonsingular complex curve germ in $X^{(\hat{\alpha})}$, we can apply slanted vector fields of low vertical pole order (from Proposition 2.19) up to as many times as the weight m of ω to get a holomorphic $(n-1)$ -jet differential $\omega^{(P_0)}$ which vanishes on an ample divisor of $X^{(\hat{\alpha})}$ and has nonzero value at P_0 .

By the Schwarz lemma of the vanishing of pullbacks of holomorphic jet differentials vanishing on ample divisor to \mathbb{C} by an entire curve, we conclude, from $\varphi^* \omega^{(P_0)} \equiv 0$ for any P_0 above y representable by nonsingular curve germ in $X^{(\hat{\alpha})}$, that y cannot be equal to $\varphi(\zeta_0)$ for some $\zeta_0 \in \mathbb{C}$ where the differential $d\varphi$ of φ is nonzero. Hence the image of any nonconstant holomorphic map φ from \mathbb{C} to $X^{(\hat{\alpha})}$ must be contained in the proper subvariety Z of $X^{(\hat{\alpha})}$.

2.22. Hyperbolicity from Existence of Appropriate Jet Differential.

In (2.21), if the holomorphic $(n-1)$ -jet differential ω on $X^{(\hat{\alpha})}$ is assumed to satisfy in addition the condition that, for some $0 < \theta'' < \theta'$, at every point of $X^{(\hat{\alpha})}$ some coefficient of ω has vanishing order no more than $e^{\theta''}$, then at any point P_0 of $J_{n-1}(X)$ representable by nonsingular complex curve germ, we can apply slanted vector fields of low vertical pole order (from Proposition 2.19) up to as many times as the weight m of ω plus $e^{\theta''}$ to get a holomorphic $(n-1)$ -jet differential $\omega^{(P_0)}$ which vanishes on an ample divisor of $X^{(\hat{\alpha})}$ and has nonzero value at P_0 . As in (2.21), we can now conclude that there is no nonconstant holomorphic map from \mathbb{C} to $X^{(\hat{\alpha})}$.

2.23. Extendibility of Jet Differentials Generic Fiber to Neighboring Fibers

In both (2.21) and (2.22) there is an assumption about the holomorphic extendibility of the jet differential on $X^{(\hat{\alpha})}$ in question to neighboring fibers $X^{(\alpha)}$. Such a condition is automatically satisfied for a generic $\hat{\alpha}$ in \mathbb{P}_N because of the following general statement.

Let $\tilde{\pi} : \mathcal{Y} \rightarrow S$ be a flat holomorphic family of compact complex spaces and $\mathcal{L} \rightarrow \mathcal{Y}$ holomorphic vector bundle. Then there exists a proper subvariety Z of S such that for $s \in S - Z$ the restriction map

$$\Gamma(U_s, \mathcal{L}) \rightarrow \Gamma(\tilde{\pi}^{-1}(s), \mathcal{L}|_{\tilde{\pi}^{-1}(s)})$$

is surjective for some open neighborhood U_s of s in S .

3. CONSTRUCTION OF HOLOMORPHIC JET DIFFERENTIALS

We are going to construct holomorphic jet differentials. One crucial ingredient is the use of the Koszul complex to show that a homogeneous polynomial of low degree in $n + 1$ homogeneous coordinates and their differentials up to order $n - 1$ cannot locally belong the ideal generated by a second homogeneous polynomial and its differentials up to order $n - 1$ when the second homogeneous polynomial is a homogeneous polynomial of high degree in the $n + 1$ homogeneous coordinates.

Lemma 3.1. *Let Y be a compact complex manifold and Z be a subvariety of pure codimension at least 2 in Y . Let \mathcal{F} be a locally free sheaf on Y . Then the restriction map*

$$H^q(Y, \mathcal{F}) \rightarrow H^q(Y - Z, \mathcal{F})$$

is an isomorphism for $0 \leq q \leq \text{codim}_Y Z - 2$.

Proof. This is a standard removability result for cohomology groups. Let $\{U_j\}_j$ be a finite cover of Y by Stein open subsets U_j . Since

$$H^r\left(\bigcap_{\ell=0}^p (U_{j_\ell} - Z), \mathcal{F}\right) = 0$$

for $1 \leq r \leq \text{codim}_Y Z - 2$ and for any j_0, \dots, j_p , by Leray's theorem the following natural isomorphism

$$H^q(Y - Z, \mathcal{F}) \approx H^q(\{U_j - Z\}_j, \mathcal{F})$$

gives the computation of the sheaf cohomology by Čech cohomology. Since the restriction map

$$\Gamma \left(\bigcap_{\ell=0}^p U_{j_\ell}, \mathcal{F} \right) \rightarrow \Gamma \left(\bigcap_{\ell=0}^p (U_{j_\ell} - Z), \mathcal{F} \right)$$

is bijective for any j_0, \dots, j_p , it follows that the map

$$H^q \left(\{U_j\}_j, \mathcal{F} \right) \approx H^q \left(\{U_j - Z\}_j, \mathcal{F} \right)$$

defined by restriction is an isomorphism. The lemma follows from the following natural isomorphism

$$H^q(Y, \mathcal{F}) \approx H^q \left(\{U_j\}_j, \mathcal{F} \right)$$

gives the computation of the sheaf cohomology by Čech cohomology. \square

Lemma 3.2. *Let ℓ and $a \leq N$ be positive integers. Let Z be a linear subspace of \mathbb{P}_N and let*

$$F_1, \dots, F_a \in \Gamma(\mathbb{P}_N, \mathcal{O}_{\mathbb{P}_N}(\ell))$$

such that the zero-set of F_1, \dots, F_a in $\mathbb{P}_N - Z$ is a submanifold of codimension a in $\mathbb{P}_N - Z$ which is a complete intersection. Assume that

$$H^q(\mathbb{P}_N - Z, \mathcal{O}_{\mathbb{P}_N}(r)) = 0 \quad \text{for } 1 \leq q < a$$

for any integer r . Then

$$\Gamma(\mathbb{P}_N - Z, \mathcal{O}_{\mathbb{P}_N}^{\oplus a}(-\ell + p)) \rightarrow \Gamma \left(\mathbb{P}_N - Z, \sum_{j=1}^a \mathcal{O}_{\mathbb{P}_N}(p) F_j \right)$$

induced by

$$(g_1, \dots, g_a) \mapsto \sum_{j=1}^a g_j F_j$$

is surjective.

Proof. Consider the Koszul complex

$$\begin{aligned} 0 \rightarrow \mathcal{O}_{\mathbb{P}_N}^{\oplus \binom{a}{a}}(-a\ell + p) \rightarrow \dots \rightarrow \mathcal{O}_{\mathbb{P}_N}^{\oplus \binom{a}{k}}(-k\ell + p) \xrightarrow{\phi_k} \mathcal{O}_{\mathbb{P}_N}^{\oplus \binom{a}{k-1}}(-(k-1)\ell + p) \\ \rightarrow \dots \rightarrow \mathcal{O}_{\mathbb{P}_N}^{\oplus \binom{a}{2}}(-2\ell + p) \rightarrow \mathcal{O}_{\mathbb{P}_N}^{\oplus \binom{a}{1}}(-\ell + p) \xrightarrow{\phi_1} \mathcal{O}_{\mathbb{P}_N}^{\oplus \binom{a}{0}}(p). \end{aligned}$$

The homomorphisms in the Koszul complex is defined as follows. Take symbols e_1, \dots, e_a . We use

$$e_{i_1} \wedge \dots \wedge e_{i_k} \quad (1 \leq i_1 < \dots < i_k \leq a)$$

as a local basis for $\mathcal{O}_{\mathbb{P}_N}^{\oplus \binom{a}{k}}(-k\ell + p)$ to represent an element

$$(g_{i_1, \dots, i_k})_{1 \leq i_1 < \dots < i_k \leq a}$$

of $\mathcal{O}_{\mathbb{P}_N}^{\oplus \binom{a}{k}}(-k\ell + p)$ as

$$\sum_{1 \leq i_1 < \dots < i_k \leq a} g_{i_1, \dots, i_k} (e_{i_1} \wedge \dots \wedge e_{i_k})$$

and define

$$\varphi_k : \mathcal{O}_{\mathbb{P}_N}^{\oplus \binom{a}{k}}(-k\ell + p) \rightarrow \mathcal{O}_{\mathbb{P}_N}^{\oplus \binom{a}{k-1}}(-(k-1)\ell + p)$$

by

$$\varphi_k(e_{i_1} \wedge \dots \wedge e_{i_k}) = \sum_{\nu=1}^k (-1)^{\nu-1} F_{i_\nu} (e_{i_1} \wedge \dots \wedge e_{i_{\nu-1}} \wedge e_{i_{\nu+1}} \wedge \dots \wedge e_{i_k})$$

in such a representation. Since the zero-set of F_1, \dots, F_a in $\mathbb{P}_N - Z$ is a submanifold of codimension a in $\mathbb{P}_N - Z$ which is a complete intersection, it follows that the Koszul complex is exact on $\mathbb{P}_N - Z$.

We are going to prove by descending induction on b for $1 \leq b \leq a-1$ that

$$H^q(\mathbb{P}_N - Z, \text{Ker } \phi_b) = 0$$

for $1 \leq q \leq b$. The case $b = a-1$ follows from the assumption of the lemma and

$$\text{Ker } \phi_{a-1} = \mathcal{O}_{\mathbb{P}_N}^{\oplus \binom{a}{a}}(-a\ell + p).$$

For $1 \leq b < a-1$ the exact sequence

$$0 \rightarrow \text{Ker } \phi_b \rightarrow \mathcal{O}_{\mathbb{P}_N}^{\oplus \binom{a}{b}}(-b\ell + p) \rightarrow \text{Ker } \phi_{b-1} \rightarrow 0$$

yields the exactness of

$$\begin{aligned} H^q \left(\mathbb{P}_N - Z, \mathcal{O}_{\mathbb{P}_N}^{\oplus \binom{a}{b}}(-b\ell + p) \right) &\rightarrow H^q(\mathbb{P}_N - Z, \text{Ker } \phi_{b-1}) \\ &\rightarrow H^{q+1}(\mathbb{P}_N - Z, \text{Ker } \phi_b) \end{aligned}$$

and for $1 \leq q \leq b-1$ we conclude, from

$$H^q \left(\mathbb{P}_N - Z, \mathcal{O}_{\mathbb{P}_N}^{\oplus \binom{a}{b}}(-b\ell + p) \right) = 0$$

in the assumption of the lemma and

$$H^{q+1}(\mathbb{P}_N - Z, \text{Ker } \phi_b) = 0$$

in the induction hypothesis that

$$H^q(\mathbb{P}_N - Z, \text{Ker } \phi_{b-1}) = 0,$$

which completes the induction argument.

For $b = 1$ we have

$$H^1(\mathbb{P}_N - Z, \text{Ker } \phi_1) = 0,$$

and the short exact sequence

$$0 \rightarrow \text{Ker } \phi_1 \rightarrow \mathcal{O}_{\mathbb{P}_N}^{\oplus a}(-\ell + p) \rightarrow \text{Im } \phi_1 \rightarrow 0$$

yields the surjectivity of

$$\Gamma(\mathbb{P}_N - Z, \mathcal{O}_{\mathbb{P}_N}^{\oplus a}(-\ell + p)) \rightarrow \Gamma(\mathbb{P}_N - Z, \text{Im } \phi_1).$$

Hence

$$\Gamma(\mathbb{P}_N - Z, \mathcal{O}_{\mathbb{P}_N}^{\oplus a}(-\ell + p)) \rightarrow \Gamma\left(\mathbb{P}_N - Z, \sum_{j=1}^a \mathcal{O}_{\mathbb{P}_N}(p) F_j\right)$$

induced by φ_1 is surjective. \square

Lemma 3.3. *Let w_0, w_1 be two transcendental variables (representing two local holomorphic functions). Then*

$$w_0^{j+1} d^j \left(\frac{w_1}{w_0} \right)$$

is a polynomial in the variables

$$d^\ell w_k \quad (0 \leq \ell \leq j, k = 0, 1)$$

which is homogeneous of degree $j + 1$ in all the variables and of total weight j in the differentials $d^\ell w_k$ for $0 \leq \ell \leq j$ and $k = 0, 1$ when the weight of $d^\ell w_k$ is assigned to be ℓ .

Proof. The case $j = 0$ of the claim is clear. The induction process of the claim going from Step j to Step $j + 1$ simply follows from

$$\begin{aligned} & w_0^{j+2} d^{j+1} \left(\frac{w_1}{w_0} \right) \\ &= w_0 \left(d \left(w_0^{j+1} d^j \left(\frac{w_1}{w_0} \right) \right) \right) - (j+1) (dw_0) \left(w_0^{j+1} d^j \left(\frac{w_1}{w_0} \right) \right) \end{aligned}$$

and the observations that

(i) the differential of a homogeneous polynomial in the variables

$$d^\ell w_k \quad (0 \leq \ell \leq j, k = 0, 1)$$

is a homogeneous polynomial in the variables

$$d^\ell w_k \quad (0 \leq \ell \leq j, k = 0, 1)$$

of the same degree, and

(ii) the differential of a polynomial in the variables

$$d^\ell w_k \quad (0 \leq \ell \leq j, k = 0, 1)$$

which is of homogeneous weight a is a polynomial in the variables

$$d^\ell w_k \quad (0 \leq \ell \leq j, k = 0, 1)$$

which is of homogeneous weight $a + 1$ when the weight of $d^\ell w_k$ is assigned to be ℓ . \square

Lemma 3.4. (Injectivity of Pullback Map for Jet Differentials) *Let $1 \leq k \leq n - 1$ and let f be a polynomial of degree δ in inhomogeneous coordinates x_1, \dots, x_n of \mathbb{P}_n so that the zero-set of f defines a complex manifold X in \mathbb{P}_n . Let Q be a non identically zero polynomial in the variables $d^j x_\ell$ ($0 \leq j \leq k, 1 \leq \ell \leq n$). Assume that Q is of degree m_0 in x_1, \dots, x_n is m_0 and is of homogeneous weight m in the variables $d^j x_\ell$ ($1 \leq j \leq k, 1 \leq \ell \leq n$) when the weight of $d^j x_\ell$ is assigned to be j . If $m_0 + 2m < \delta$, then Q is not identically zero on the space of k -jets of X .*

Proof. Suppose Q is identically zero on the space of k -jets of X . We are going to derive a contradiction.

Since Q is of homogeneous weight m in the variables $d^j x_\ell$ ($1 \leq j \leq k, 1 \leq \ell \leq n$) when the weight of $d^j x_\ell$ is assigned to be j , it follows that the degree of Q in the variables $d^j x_\ell$ ($1 \leq j \leq k, 1 \leq \ell \leq n$) is at most m . We introduce the homogeneous coordinates

$$z_0, z_1, \dots, z_n$$

of \mathbb{P}_n so that

$$x_j = \frac{z_j}{z_0} \quad (1 \leq j \leq n).$$

Let $N = (k + 1)(n + 1) - 1$ and relabel the variables

$$d^j z_\ell \quad (0 \leq j \leq k, 0 \leq \ell \leq n)$$

as the $N + 1$ homogeneous coordinates w_0, \dots, w_N of \mathbb{P}_N . Let $P = z_0^{m_0+2m} Q$. Since the degree of Q in the variables $d^j x_\ell$ ($1 \leq j \leq k, 1 \leq \ell \leq n$) is at most m , by Lemma 3.3 we conclude that P is a polynomial in the variables w_0, \dots, w_N and is homogeneous of degree $m_0 + 2m$.

We are going to apply Lemma 3.2. In our application We set $m = n$. The homogeneous polynomials F_1, \dots, F_m of degree δ in the $N + 1$ homogeneous coordinates w_0, \dots, w_N of \mathbb{P}_N are

$$z_0^\delta f, d(z_0^\delta f), \dots, d^k(z_0^\delta f).$$

The linear subspace Z in \mathbb{P}_N is defined by $z_0 = z_1 = \cdots = z_n = 0$ which is of complex codimension $n+1$ in \mathbb{P}_N and is therefore of complex dimension $N - (n+1) = k(n+1) - 1$.

We know that, if \hat{Z} is a subvariety of \mathbb{P}_N , then for any Stein open subset U of \mathbb{P}_N the cohomology group $H^q(U - \hat{Z}, \mathcal{O}_{\mathbb{P}_N})$ vanishes for $0 \leq q \leq \text{codim}_{\mathbb{P}_N} \hat{Z} - 2 = n - 1$, where $\text{codim}_{\mathbb{P}_N} \hat{Z}$ means the complex codimension of Z in \mathbb{P}_N . Thus,

$$H^q(\mathbb{P}_N - Z, \mathcal{O}_{\mathbb{P}_N}(\ell)) = 0 \quad \text{for } 0 \leq q \leq n - 1.$$

Since Q is identically zero on the space of k -jets of X , it follows that P locally belongs to the ideal generated by

$$z_0^\delta f, d(z_0^\delta f), \dots, d^k(z_0^\delta f).$$

By Lemma 3.2 with $a = k + 1$, since

$$\text{codim}_{\mathbb{P}_N} Z - 2 = n - 1 \geq k = a - 1,$$

we can write

$$P = \sum_{j=0}^k g_j d^j(z_0^\delta f)$$

for some homogeneous polynomials g_0, \dots, g_k of the variables

$$w_0, \dots, w_N,$$

where the total degree of g_j is $m_0 + 2m - \delta$. We arrive at a contradiction, because $m_0 + 2m - \delta$ is negative and a polynomial cannot have a negative degree. \square

Now we count the number of unknowns and the number of equations.

Lemma 3.5. *Let X be a hypersurface of degree δ in \mathbb{P}_n . Let S be a hypersurface in X defined by a homogeneous polynomial g of degree s in the homogeneous coordinates of \mathbb{P}_n . Then for $q \geq \delta + s + n$,*

$$\dim \Gamma(S, \mathcal{O}_S(q)) = \sum_{j=1}^{\delta} \sum_{k=1}^s \binom{n+q-j-k}{n-2}.$$

In particular,

$$\dim \Gamma(S, \mathcal{O}_S(q)) \leq \frac{s \delta (n+q-2)^{n-2}}{(n-2)!}$$

for $q \geq \delta + s + n$.

Proof. First of all, for any nonnegative integer ℓ we have

$$\dim \Gamma(\mathbb{P}_n, \mathcal{O}_{\mathbb{P}_n}(\ell)) = \binom{\ell+n}{\ell} = \binom{\ell+n}{n},$$

because it is equal to the number of possibilities of choosing ℓ elements out of $n+1$ elements with repetition allowed which is the same as choosing ℓ elements out of $n+1+\ell-1 = \ell+n$ elements without repetition. From the exact sequence

$$(3.5.1) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}_n}(\ell) \xrightarrow{\phi_f} \mathcal{O}_{\mathbb{P}_n}(\ell+\delta) \rightarrow \mathcal{O}_X(\ell+\delta) \rightarrow 0$$

where ϕ_f is defined by multiplication by f , it follows that

$$\begin{aligned} \Gamma(\mathbb{P}_n, \mathcal{O}_{\mathbb{P}_n}(\ell)) &\rightarrow \Gamma(\mathbb{P}_n, \mathcal{O}_{\mathbb{P}_n}(\ell+\delta)) \rightarrow \\ \Gamma(X, \mathcal{O}_X(\ell+\delta)) &\rightarrow H^1(\mathbb{P}_n, \mathcal{O}_{\mathbb{P}_n}(\ell)) = 0 \end{aligned}$$

is exact and

$$\Gamma(X, \mathcal{O}_X(\ell+\delta)) = \Gamma(\mathbb{P}_n, \mathcal{O}_{\mathbb{P}_n}(\ell+\delta)) / f\Gamma(\mathbb{P}_n, \mathcal{O}_{\mathbb{P}_n}(\ell)).$$

Hence

$$\dim \Gamma(X, \mathcal{O}_X(\ell+\delta)) = \binom{n+\ell+\delta}{n} - \binom{n+\ell}{n}$$

From (3.5.1) we have the exact sequence

$$H^p(\mathbb{P}_n, \mathcal{O}_{\mathbb{P}_n}(\ell+\delta)) \rightarrow H^p(X, \mathcal{O}_X(\ell+\delta)) \rightarrow H^{p+1}(\mathbb{P}_n, \mathcal{O}_{\mathbb{P}_n}(\ell)).$$

From the vanishing of $H^p(\mathbb{P}_n, \mathcal{O}_{\mathbb{P}_n}(\ell+\delta))$ for $1 \leq p < n$ it follows that $H^p(X, \mathcal{O}_X(\ell+\delta)) = 0$ for $1 \leq p < n-1$. From the exact sequence

$$0 \rightarrow \mathcal{O}_X(\ell) \xrightarrow{\phi_g} \mathcal{O}_X(\ell+s) \rightarrow \mathcal{O}_S(\ell+s) \rightarrow 0$$

where ϕ_g is defined by multiplication by g and from

$$H^1(X, \mathcal{O}_X(\ell)) = 0$$

for $n \geq 3$ it follows that

$$\Gamma(S, \mathcal{O}_S(\ell+s)) = \Gamma(X, \mathcal{O}_X(\ell+s)) / g\Gamma(X, \mathcal{O}_X(\ell)).$$

Hence

$$\begin{aligned} \dim \Gamma(S, \mathcal{O}_S(\ell+s)) &= \dim \Gamma(X, \mathcal{O}_X(\ell+s)) - \dim \Gamma(X, \mathcal{O}_X(\ell)) \\ &= \left[\binom{n+\ell+s}{n} - \binom{n+\ell+s-\delta}{n} \right] - \left[\binom{n+\ell}{n} - \binom{n+\ell-\delta}{n} \right] \end{aligned}$$

for $\ell \geq \delta$. We are going to use the following identity for binomial coefficients

$$\binom{a}{b} - \binom{a-1}{b} = \binom{a}{b-1}$$

for $a - 1 \geq b \geq 1$. Then for $q \geq \delta + s + n$, we have

$$\begin{aligned}
\dim \Gamma(S, \mathcal{O}_S(q)) &= \left[\binom{n+q}{n} - \binom{n+q-\delta}{n} \right] \\
&\quad - \left[\binom{n+q-s}{n} - \binom{n+q-s-\delta}{n} \right] \\
&= \sum_{j=1}^{\delta} \left[\binom{n+q-j+1}{n} - \binom{n+q-j}{n} \right] \\
&\quad - \sum_{j=1}^q \left[\binom{n+q-s-j+1}{n} - \binom{n+q-s-j}{n} \right] \\
&= \sum_{j=1}^{\delta} \binom{n+q-j}{n-1} - \sum_{j=1}^q \binom{n+q-s-j}{n-1} \\
&= \sum_{j=1}^{\delta} \left[\binom{n+q-j}{n-1} - \binom{n+q-s-j}{n-1} \right] \\
&= \sum_{j=1}^{\delta} \sum_{k=1}^s \left[\binom{n+q-j-k+1}{n-1} - \binom{n+q-j-k}{n-1} \right] \\
&= \sum_{j=1}^{\delta} \sum_{k=1}^s \binom{n+q-j-k}{n-2}.
\end{aligned}$$

□

Lemma 3.6. *Let y_1, \dots, y_r be independent transcendental variables. Let $1 = n_1 \leq n_2 \leq \dots \leq n_r$ be integers. Let a_m be the number of all monomials $y_1^{k_1} \dots y_r^{k_r}$ such that $\sum_{j=1}^r n_j k_j = m$. Let a_m be the number of elements in A_m . Then*

$$\left(\left\lfloor \frac{m}{n_r} \right\rfloor + r - 1 \right) \leq a_m \leq \binom{m+r-1}{r-1},$$

where $\lfloor u \rfloor$ denotes the largest integer not exceeding u .

Proof. Let A_m be the set of all monomials $y_2^{k_2} \dots y_r^{k_r}$ such that $\sum_{j=2}^r n_j k_j = m$. Since $n_1 = 1$, A_m is the same as the set of all monomials $y_1^{k_1} \dots y_r^{k_r}$ such that $\sum_{j=1}^r n_j k_j = m$ and a_m is the number of elements of A_m . Let B_m be the set of all monomials $y_2^{k_2} \dots y_r^{k_r}$ such that $\sum_{j=2}^r k_j \leq \left\lfloor \frac{m}{n_1} \right\rfloor$. Let C_m be the set of all monomials $y_2^{k_2} \dots y_r^{k_r}$ such that $\sum_{j=1}^r k_j \leq m$.

Since

$$\begin{aligned} \sum_{j=2}^r k_j &\leq \left\lfloor \frac{m}{n_r} \right\rfloor \\ \implies \sum_{j=2}^r n_j k_j &\leq m \\ \implies \sum_{j=2}^r k_j &\leq m, \end{aligned}$$

it follows that

$$B_m \subset A_m \subset C_m.$$

Since the number of elements in B_m is

$$\binom{\left\lfloor \frac{m}{n_r} \right\rfloor + r - 1}{r - 1}$$

and the number of elements in C_m is

$$\binom{m + r - 1}{r - 1},$$

the conclusion of the Lemma follows. \square

Lemma 3.7. *Let f be a polynomial of degree δ in the variables x_1, \dots, x_n . Let κ_ℓ be the smallest nonnegative integer such that $(f_{x_1})^{\kappa_\ell} d^\ell x_1$ can be expressed as a polynomial P_ℓ of*

$$x_1, x_2, \dots, x_n, \\ d^j x_r \quad (1 \leq j \leq \ell, 2 \leq r \leq n)$$

on the space of ℓ -jets of the zero-set of f . Then $\kappa_1 = 1$ and

$$\kappa_\ell \leq 1 + \max \left\{ \sum_{j=1}^{\ell-1} \kappa_j s_j \mid s_1 + 2s_2 + \dots + (\ell-1)s_{\ell-1} \leq \ell \right\}.$$

Moreover, as a polynomial of the degree of P_ℓ in the variables x_1, \dots, x_n is at most $\kappa_\ell(\delta - 1)$. The integers κ_ℓ in can be estimated by $\kappa_\ell \leq \ell!$

Proof. We use induction on ℓ for $\ell \geq 1$. The case $\ell = 1$ is clear, because

$$f_{x_1} dx_1 = - \sum_{r=2}^n f_{x_r} dx_r$$

and we can set

$$P_1 = - \sum_{r=2}^n f_{x_r} dx_r$$

whose degree in x_1, \dots, x_r is obviously at most $\delta - 1$. From $d^\ell f = 0$ it follows that on the space of ℓ -jets of the zero-set of f the jet differential $f_{x_1} d^\ell x_1$ can be written as a polynomial Q_ℓ in

$$d^j x_r \quad (1 \leq j \leq \ell - 1, 1 \leq r \leq n), \\ d^\ell x_2, \dots, d^\ell x_n$$

of weight $\leq \ell$ in

$$d^j x_r \quad (2 \leq j \leq \ell - 1, 1 \leq r \leq n), \\ d^\ell x_2, \dots, d^\ell x_n$$

when $d^j x_r$ is given the weight j . As a polynomial in $dx_1, \dots, d^{\ell-1} x_1$ the total weight of Q_ℓ is no more than ℓ . As a polynomial in x_1, \dots, x_n the degree of Q_ℓ is at most $\delta - 1$. Thus inductively on ℓ we conclude that we need only to multiply $f_{x_1} d^\ell x_1$ by a power of f_{x_1} not exceeding

$$\max \left\{ \sum_{j=1}^{\ell-1} \kappa_j s_j \mid s_1 + 2s_2 + \dots + (\ell-1)s_{\ell-1} \leq \ell \right\}$$

to yield a polynomial P_ℓ of

$$x_1, x_2, \dots, x_n, \\ d^j x_r \quad (1 \leq j \leq \ell, 2 \leq r \leq n)$$

on the space of ℓ -jets of the zero-set of f . Moreover, the degree of P_ℓ in x_1, \dots, x_n is at most $\kappa_\ell (\delta - 1)$.

The integers κ_ℓ can be estimated by $\kappa_\ell \leq \ell!$, because, when

$$r_1 + 2r_2 + \dots + (\ell-1)r_{\ell-1} \leq \ell,$$

we have $r_j \leq \frac{\ell}{j}$, and from $\kappa_j \leq j!$ for $1 \leq j \leq \ell - 1$ it follows that

$$\sum_{j=1}^{\ell-1} \kappa_j r_j \leq \sum_{j=1}^{\ell-1} \kappa_j \left(\frac{\ell}{j} \right) \leq \ell \sum_{j=1}^{\ell-1} (j-1)! < \ell \sum_{j=1}^{\ell-1} (\ell-2)! = \ell!.$$

□

Proposition 3.8. (Jet Differential from Polynomial in Differentials of Inhomogeneous Coordinates) *Let X be a nonsingular hypersurface of degree δ in \mathbb{P}_n defined by a polynomial $f(x_1, \dots, x_n)$ of degree δ in the affine coordinates x_1, \dots, x_n of \mathbb{P}_n . Suppose $\epsilon, \epsilon', \theta_0, \theta$, and θ' are numbers in the open interval $(0, 1)$ such that $n\theta_0 + \theta \geq n + \epsilon$ and $\theta' < 1 - \epsilon'$. Then there exists an explicit positive number $A = A(n, \epsilon, \epsilon')$ depending only on n, ϵ , and ϵ' such that for $\delta \geq A$ and any nonsingular hypersurface X in \mathbb{P}_n of degree δ there exists a non identically zero $\mathcal{O}_{\mathbb{P}_n}(-q)$ -valued holomorphic $(n-1)$ -jet differential ω on X of total*

weight m with $q \geq \delta^{\theta'}$ and $m \leq \delta^{\theta}$. Here, with respect to a local holomorphic coordinate system w_1, \dots, w_{n-1} of X , the weight of ω is in the variables $d^j w_\ell$ ($1 \leq j \leq n-1, 1 \leq \ell \leq n-1$) with the weight j assigned to $d^j w_\ell$. Moreover, for any affine coordinates x_1, \dots, x_n of \mathbb{P}_n , when $f_{x_1} = 1$ defines in a nonsingular hypersurface in X , the $(n-1)$ -jet differential ω can be chosen to be of the form $\frac{Q}{f_{x_1}-1}$, where Q is a polynomial in

$$d^j x_1, \dots, d^j x_n \quad (0 \leq j \leq n-1)$$

which is of degree $m_0 = \lceil \delta^{\theta_0} \rceil$ in x_1, \dots, x_n and is of homogeneous weight $m = \lceil \delta^{\theta} \rceil$ in

$$d^j x_1, \dots, d^j x_n \quad (1 \leq j \leq n-1)$$

when the weight of $d^j x_\ell$ is assigned to be j .

Proof. Let x_1, \dots, x_n and z_0, \dots, z_n be respectively the homogeneous and inhomogeneous coordinates of \mathbb{P}_n so that $x_j = \frac{z_j}{z_0}$ for $1 \leq j \leq n$. Let f be a polynomial of degree δ in x_1, \dots, x_n so that the zero-set of f in \mathbb{P}_n is X .

Consider a non identically zero polynomial Q in

$$d^j x_1, \dots, d^j x_n \quad (0 \leq j \leq n-1)$$

which is of degree m_0 in x_1, \dots, x_n and is of homogeneous weight m in

$$d^j x_1, \dots, d^j x_n \quad (1 \leq j \leq n-1)$$

when the weight of $d^j x_\ell$ is assigned to be j . We impose the condition

$$m_0 + 2m < \delta$$

so that according to Lemma 3.4 the pullback of Q to the space of $(n-1)$ -jets of $\{f=0\}$ is not identically zero. According to Lemma 3.6 the degree of freedom in the choice of the polynomial Q is at least

$$\binom{m_0 + n}{n} \binom{\lfloor \frac{m}{n-1} \rfloor + n(n-1) - 1}{n(n-1) - 1},$$

where the first factor

$$\binom{m_0 + n}{n}$$

is the number of monomials of degree $\leq m_0$ in n variables x_1, \dots, x_n and the second factor

$$\binom{\lfloor \frac{m}{n-1} \rfloor + n(n-1) - 1}{n(n-1) - 1},$$

is the number of monomials of *homogeneous* degree $\lfloor \frac{m}{n-1} \rfloor$ in the $n(n-1)$ variables

$$d^j x_\ell \quad (1 \leq j \leq n-1, 1 \leq \ell \leq n).$$

The key point of this proof is that though we have all the variables $d^j x_1, \dots, d^j x_n$ for $0 \leq j \leq n-1$, we do not have to worry about the dependence resulting from the relations $d^j f = 0$ ($0 \leq j \leq n-1$).

Let $H_{\mathbb{P}_n}$ be the hyperplane of \mathbb{P}_n defined by $x_n = 0$. We now want the meromorphic $(n-1)$ -jet differential defined by Q to be holomorphic on X and and, moreover, to vanish at $X \cap H_{\mathbb{P}_n}$ of order q . First of all, on the space of $(n-1)$ -jets of X , we can use the relation $d^j f = 0$ ($1 \leq j \leq n-1$) to eliminate the variables $d^j x_1$ ($1 \leq j \leq n-1$) by expressing $d^j x_1$ ($1 \leq j \leq n-1$) in terms of

$$d^j x_\ell \quad (1 \leq j \leq n-1, 2 \leq \ell \leq n).$$

To do this, according to Lemma 3.7 we can multiply Q by $(f_{x_1})^{\tilde{N}}$ with $\tilde{N} = 2m \sum_{j=1}^{n-1} \kappa_j$, because, in a monomial of weight m , the degree of $d^j x_\ell$ ($1 \leq j \leq n-1, 1 \leq \ell \leq n$) is at most $\lfloor \frac{m}{j} \rfloor$ and $(j+1) \lfloor \frac{m}{j} \rfloor \leq 2m$. Since $\kappa_j \leq j!$, it follows that $\tilde{N} \leq (n-1)!2m$. Let $N = (n-1)!2m$. The degree of $(f_{x_1})^N Q$ in x_1, \dots, x_n is now $m_0 + N(\delta-1)$ and the weight of $(f_{x_1})^N Q$ in $d^j x_\ell$ ($1 \leq j \leq n-1, 2 \leq \ell \leq n$) is homogeneous and equal to m when the weight of $d^j x_\ell$ is assigned to be j .

We let S be the divisor in X defined by $f_{x_1} - 1 = 0$. The hypersurface in \mathbb{P}_n defined by $f_{x_1} - 1 = 0$ is of degree $\delta - 1$. We observe that for a generic polynomial f of degree δ , the divisor S in X is nonsingular, because it is the case when f equals the Fermat hypersurface

$$F = \sum_{j=1}^n x_j^\delta - 1.$$

Then

$$F_{x_1} = \delta x_1^{\delta-1}$$

and the $2 \times n$ matrix

$$\begin{bmatrix} x_1^{\delta-1} & x_2^{\delta-1} & \cdots & x_n^{\delta-1} \\ x_1^{\delta-2} & 0 & \cdots & 0 \end{bmatrix}$$

whose rows are nonzero multiplies of the gradients of F and F_{x_1} has rank 2 unless either $x_1 = 0$ or $x_2 = \dots = x_n = 0$, which is impossible, because on S one has $|x_1| = \delta^{\frac{-1}{\delta-1}}$ from $F_{x_1} = 1$ and the condition $x_2 = \dots = x_n = 0$ implies $|x_1| = 1 \neq \delta^{\frac{-1}{\delta-1}}$ when $F = 0$. To prove this Lemma we need only prove it for a generic f and then remove

the genericity assumption for f by using the semi-continuity of the dimension of the space of global holomorphic sections over a fiber in a holomorphic family of compact complex manifolds and a holomorphic vector bundle.

The polynomial Q defines a meromorphic $(n-1)$ -jet differential on X which we again denote by Q . We now count the pole order of the jet differential Q on X at $X \cap H_{\mathbb{P}_n}$. For the counting of this pole order, we introduce another set of inhomogeneous coordinates ζ_1, \dots, ζ_n of \mathbb{P}_n defined by

$$\zeta_1 = \frac{x_1}{x_n}, \dots, \zeta_{n-1} = \frac{x_{n-1}}{x_n}, \zeta_n = \frac{1}{x_n}$$

so that

$$x_1 = \frac{\zeta_1}{\zeta_n}, \dots, x_{n-1} = \frac{\zeta_{n-1}}{\zeta_n}, x_n = \frac{1}{\zeta_n}.$$

Since by Lemma 3.3

$$\zeta_n^{j+1} d^j x_\ell = \zeta_n^{j+1} d^j \left(\frac{\zeta_\ell}{\zeta_n} \right) \quad (1 \leq \ell \leq n-1)$$

and

$$\zeta_n^{j+1} d^j x_n = \zeta_n^{j+1} d^j \left(\frac{1}{\zeta_n} \right)$$

are polynomials in $d^k \zeta_r$ ($0 \leq k \leq j, 1 \leq r \leq n$). Thus $\zeta_n^{2m} Q$ is a polynomial in $d^k \zeta_r$ ($0 \leq k \leq j, 1 \leq r \leq n$). The pole order of the jet differential Q at $X \cap H_{\mathbb{P}_n}$ is at most $m_0 + 2m$.

We are going to show that we can choose the coefficients of the polynomial Q so that the $(n-1)$ -jet differential Q is zero at points of S . This would imply that the $(n-1)$ -jet differential

$$\frac{1}{f_{x_1} - 1} Q$$

is holomorphic on X and vanishes to order $\delta - m_0 - 2m$ at $X \cap H_{\mathbb{P}_n}$. The reason is the following. For some proper subvariety Z of $X \cap H_{\mathbb{P}_n}$ the function $\zeta_0^{\delta-1} (f_{x_1} - 1)$ is holomorphic and nowhere at points of $X \cap H_{\mathbb{P}_n} - Z$. Thus

$$\frac{1}{f_{x_1} - 1} Q = \frac{1}{\zeta_0^{\delta-1} (f_{x_1} - 1)} (\zeta_0^{\delta-1-2m}) (\zeta_0^{2m} Q)$$

is holomorphic on $X - Z$ and vanishes to order at least $\delta - 1 - 2m$ along $X \cap H_{\mathbb{P}_n} - Z$. What we want follows from Hartogs' extension theorem because Z is of complex codimension at least 2 in X .

Now on $J_{n-1}(X) \mid S$ (which is the part of the space of $(n-1)$ -jets of X lying over S) the jet differential Q equals to the jet differential

$$(f_{x_1})^N Q$$

because $f_{x_1} = 1$ holds on S . Since the degree of $(f_{x_1})^N Q$ in x_1, \dots, x_n is $m_0 + N(\delta - 1)$ and the weight of $(f_{x_1})^N Q$ in $d^k z_r$ ($1 \leq k \leq j, 2 \leq r \leq n$) is homogeneous and equal to $2m$, it follows from Lemma 3.5 that the number of linear equations, with the coefficients of Q as unknowns, needed for $(f_{x_1})^N Q$ to vanish at all points S is no more than the product

$$\frac{(\delta - 1) \delta (m_0 + N(\delta - 1))^{n-2}}{(n-2)!} \binom{m + (n-1)^2 - 1}{(n-1)^2 - 1}.$$

For the existence of a nontrivial Q with the required vanishing at all points of S , it suffices to have

$$\begin{aligned} & \binom{m_0 + n}{n} \binom{\lfloor \frac{m}{n-1} \rfloor + n(n-1) - 1}{n(n-1) - 1} \\ & > \frac{(\delta - 1) \delta (m_0 + N(\delta - 1))^{n-2}}{(n-2)!} \binom{m + (n-1)^2 - 1}{(n-1)^2 - 1}. \end{aligned}$$

In particular, it suffices that

$$\frac{(m_0 + 1)^n \left(\frac{m}{n-1}\right)^{n(n-1)-1}}{n! (n(n-1) - 1)!}$$

is greater than

$$\frac{[(\delta - 1) \delta (m_0 + (n-1)! 2m (\delta - 1))^{n-2}] [(m + (n-1)^2 - 1)^{(n-1)^2 - 1}]}{(n-2)! ((n-1)^2 - 1)!}.$$

We choose $m_0 = \lceil \delta^{\theta_0} \rceil$, $m = \lceil \delta^\theta \rceil$, and $q = \lfloor \delta^{\theta'} \rfloor$. Since the three positive numbers θ_0 , θ , and θ' are all strictly less than 1, measured in terms of powers of δ as δ becomes dominantly large, the order of

$$\frac{(m_0 + 1)^n \left(\frac{m}{n-1}\right)^{n(n-1)-1}}{n! (n(n-1) - 1)!}$$

is at least

$$\delta^{n\theta_0 + (n(n-1)-1)\theta}$$

and the order of

$$\frac{[(\delta - 1) \delta (m_0 + (n-1)! 2m (\delta - 1))^{n-2}] [(m + (n-1)^2 - 1)^{(n-1)^2 - 1}]}{(n-2)! ((n-1)^2 - 1)!}$$

is at most

$$\delta^{2+(n-2)(1+\theta)+((n-1)^2-1)\theta}.$$

Since by assumption $n\theta_0 + \theta \geq n + \epsilon$, it follows that

$$[n\theta_0 + (n(n-1) - 1)\theta] - [2 + (n-2)(1+\theta) + ((n-1)^2 - 1)\theta] \geq \epsilon.$$

So there exists a positive number A depending only on n and ϵ such that

$$\frac{(m_0 + 1)^n \left(\frac{m}{n-1}\right)^{n(n-1)-1}}{n! (n(n-1) - 1)!}$$

is greater than

$$\frac{[(\delta - 1)\delta(m_0 + (n-1)!2m(\delta - 1))^{n-2}] [(m + (n-1)^2 - 1)^{(n-1)^2-1}]}{(n-2)!((n-1)^2 - 1)!}$$

when $\delta \geq A$. We can also assume that A is chosen so that $\delta - \delta^{\theta_0} - 2\delta^{\theta} \geq \delta^{\theta'}$ for $\delta \geq A$ to make sure that the $(n-1)$ -jet differential

$$\frac{1}{f_{x_1} - 1} Q$$

is holomorphic on X and vanishes to order at least q at $X \cap H_{\mathbb{P}_n}$. \square

Remark 3.9. Proposition 3.8 is the same as Proposition 4.6 on p.446 of [Si02] and also the same as Proposition 2 on p.558 of [Si04].

4. HYPERBOLICITY FROM SLANTED VECTOR FIELDS AND VANISHING ORDER OF LOW-DEGREE POLYNOMIALS

Since the holomorphic jet differential $\omega^{(\alpha)}$ vanishing on an ample divisor of $X^{(\alpha)}$ from the construction in Proposition 3.8 is of the form $\frac{Q^{(\alpha)}}{f_{x_1}^{(\alpha)} - 1}$, the discussion in (2.21) and (2.22) shows that the hyperbolicity of generic hypersurface of high degree is now reduced to the control of the vanishing order of the pullback to $X^{(\alpha)}$ of a polynomial of low degree in $d^\ell x_j$ for $0 \leq \ell \leq n-1$ and $1 \leq j \leq n$. In this section we deal with this question of the control of the vanishing order of such pullbacks and finish the proofs of Theorem 0.1 and Theorem 0.2.

Lemma 4.1. (VANISHING ORDER OF RESTRICTION OF POLYNOMIALS) *Let W be a nonempty connected open subset of \mathbb{P}_N such that $X^{(\alpha)}$ is regular for every $\alpha \in W$. Let m be a positive integer and for $\alpha \in W$ let $g^{(\alpha)}$ be a homogeneous polynomial of degree m in the homogeneous coordinates z_0, \dots, z_n of \mathbb{P}_n whose coefficients $g_{\mu_0, \dots, \mu_n}^{(\alpha)}$ (for $\mu_0 + \dots + \mu_n = m$) are holomorphic functions of α on W , which is*

not identically zero on \mathbb{P}_n for $\alpha \in W$. Let $\sigma(\alpha)$ be a holomorphic local section of \mathcal{X} over W . Then there is a proper subvariety $\tilde{\mathcal{Z}}_{g,\sigma}$ of W such that for $\alpha \in W - \tilde{\mathcal{Z}}_{g,\sigma}$ the vanishing order of the restriction $g^{(\alpha)}|_{X^{(\alpha)}}$ of $g^{(\alpha)}$ to $X^{(\alpha)}$ at the point $\sigma(\alpha)$ of $X^{(\alpha)}$ is no more than m . As a consequence, for any relatively compact nonempty subset \tilde{W} of W there exists a proper subvariety $\tilde{\mathcal{Z}}_g$ of \tilde{W} such that for $\alpha \in \tilde{W} - \tilde{\mathcal{Z}}_g$ the vanishing order of the restriction $g^{(\alpha)}|_{X^{(\alpha)}}$ of $g^{(\alpha)}$ to $X^{(\alpha)}$ at any point of $X^{(\alpha)}$ is no more than m .

Proof. We prove first the first conclusion involving the local holomorphic section σ of \mathcal{X} over W and then prove the second statement without the local holomorphic section σ of \mathcal{X} over W later. Since the set of all $\alpha \in W$ such that the vanishing order of the restriction $g^{(\alpha)}|_{X^{(\alpha)}}$ of $g^{(\alpha)}$ to $X^{(\alpha)}$ at the point $\sigma(\alpha)$ of $X^{(\alpha)}$ is $> m$ is a subvariety of W which may be all of W , it suffices to show that there exists some point $\hat{\alpha}$ of W such that the vanishing order of the restriction $g^{(\hat{\alpha})}|_{X^{(\hat{\alpha})}}$ of $g^{(\hat{\alpha})}$ to $X^{(\hat{\alpha})}$ at the point $\sigma(\hat{\alpha})$ of $X^{(\hat{\alpha})}$ is no more than m . We pick some point α^* of W and for notational simplicity, when we need to replace W by a smaller open neighborhood W' of α^* in W , we use the same symbol W for the new smaller neighborhood W' . When we need to replace α^* by some other point, we use the same symbol α^* for the new point.

We now make the reduction to the special case that $\sigma(\alpha)$ is always the origin in some inhomogeneous coordinates system of \mathbb{P}_n for $\alpha \in W$. Choose an inhomogeneous coordinate system x_1, \dots, x_n of \mathbb{P}_n centered at $\sigma(\alpha^*)$ and let $(\sigma_1(\alpha), \dots, \sigma_n(\alpha))$ be the coordinates of $\sigma(\alpha)$ with respect to x_1, \dots, x_n . We can assume without loss of generality that $(\sigma_1(\alpha), \dots, \sigma_n(\alpha))$ belongs to the affine part \mathbb{C}^n of \mathbb{P}_n for $\alpha \in W$ (after replacing W by a smaller open neighborhood of α^* in W if necessary). For $\alpha \in W$ let

$$g^{(\alpha)} = \sum_{\nu_0 + \dots + \nu_n = m} g_{\nu_0, \dots, \nu_n}^{(\alpha)} x_1^{\nu_1} \dots x_n^{\nu_n}.$$

We introduce the translation of the inhomogeneous coordinates $y_j = x_j - \sigma_j(\alpha)$ for $1 \leq j \leq n$ and $\alpha \in W$ to write

$$\begin{aligned} g^{(\alpha)} &= \sum_{\nu_0 + \dots + \nu_n = m} g_{\nu_0, \dots, \nu_n}^{(\alpha)} (y_1 + \sigma_1(\alpha))^{\nu_1} \dots (y_n + \sigma_n(\alpha))^{\nu_n} \\ &= \sum_{\nu_0 + \dots + \nu_n = m} \hat{g}_{\nu_0, \dots, \nu_n}^{(\alpha)} y_1^{\nu_1} \dots y_n^{\nu_n} \end{aligned}$$

and

$$\begin{aligned} f^{(\alpha)} &= \sum_{\nu_0 + \dots + \nu_n = \delta} \alpha_{\nu_0, \dots, \nu_n} (y_1 + \sigma_1(\alpha))^{\nu_1} \cdots (y_n + \sigma_n(\alpha))^{\nu_n} \\ &= \sum_{\nu_0 + \dots + \nu_n = \delta} \gamma_{\nu_0, \dots, \nu_n} y_1^{\nu_1} \cdots y_n^{\nu_n}. \end{aligned}$$

Let $\gamma = [\gamma_{\nu_0, \dots, \nu_n}]_{\nu_0 + \dots + \nu_n = \delta}$. Let \tilde{W} be the open subset of \mathbb{P}_N which is the image of W under the map $\alpha \mapsto \gamma$. Let $\theta(\gamma)$ be the inverse function of $\alpha \mapsto \gamma$. The function

$$\sum_{\nu_0 + \dots + \nu_n = m} \hat{g}_{\nu_0, \dots, \nu_n}^{(\theta(\gamma))} y_1^{\nu_1} \cdots y_n^{\nu_n}$$

on the zero-set of the function

$$\sum_{\nu_0 + \dots + \nu_n = \delta} \gamma_{\nu_0, \dots, \nu_n} y_1^{\nu_1} \cdots y_n^{\nu_n}$$

near the point $y_1 = \dots = y_n = 0$ is the same as the function

$$\sum_{\nu_0 + \dots + \nu_n = m} g_{\nu_0, \dots, \nu_n}^{(\alpha)} x_1^{\nu_1} \cdots x_n^{\nu_n}$$

on the zero-set of the function

$$\sum_{\nu_0 + \dots + \nu_n = \delta} \alpha_{\nu_0, \dots, \nu_n} x_1^{\nu_1} \cdots x_n^{\nu_n}$$

near the point given by $x_j = \sigma_j(\alpha)$ for $1 \leq j \leq n$, because of the coordinates transformation $y_j = x_j - \sigma_j(\alpha)$ for $1 \leq j \leq n$. Define

$$\tilde{g}^{(\alpha)} = \sum_{\nu_0 + \dots + \nu_n = m} \tilde{g}_{\nu_0, \dots, \nu_n}^{(\alpha)} x_1^{\nu_1} \cdots x_n^{\nu_n},$$

with $\tilde{g}_{\nu_0, \dots, \nu_n}^{(\alpha)} = \hat{g}_{\nu_0, \dots, \nu_n}^{(\theta(\alpha))}$. With the change of symbols from y_j to x_j and from γ to α , the function

$$\sum_{\nu_0 + \dots + \nu_n = m} \hat{g}_{\nu_0, \dots, \nu_n}^{(\theta(\gamma))} y_1^{\nu_1} \cdots y_n^{\nu_n}$$

on the zero-set of the function

$$\sum_{\nu_0 + \dots + \nu_n = \delta} \gamma_{\nu_0, \dots, \nu_n} y_1^{\nu_1} \cdots y_n^{\nu_n}$$

near the point $y_1 = \dots = y_n = 0$ is the same as the function

$$\sum_{\nu_0 + \dots + \nu_n = m} \hat{g}_{\nu_0, \dots, \nu_n}^{(\theta(\alpha))} x_1^{\nu_1} \cdots x_n^{\nu_n}$$

on the zero-set of the function

$$\sum_{\nu_0 + \dots + \nu_n = \delta} \alpha_{\nu_0, \dots, \nu_n} x_1^{\nu_1} \cdots x_n^{\nu_n}$$

near the point $x_1 = \dots = x_n = 0$. Thus, by replacing $g^{(\alpha)}$ by $\tilde{g}^{(\alpha)}$ and replacing W by \tilde{W} , we can reduce the general case to the special case of $\sigma(\alpha)$ equal to the origin $x_1 = \dots = x_n = 0$ for all $\alpha \in W$.

Now we can assume without loss of generality that $\sigma(\alpha)$ equal to the origin $x_1 = \dots = x_n = 0$ for all $\alpha \in W$. Let p be the minimum vanishing order of $g^{(\alpha)}$ on \mathbb{P}_n at $x_1 = \dots = x_n = 0$ for $\alpha \in W$, which is achieved at some point α^* of W . We have $0 \leq p \leq m$, because the vanishing order p is for $g^{(\alpha^*)}$ on \mathbb{P}_n and *not* for the restriction of $g^{(\alpha^*)}$ on $X^{(\alpha^*)}$. If $p = 0$, then $g^{(\alpha^*)}$ is nonzero at the origin $x_1 = \dots = x_n = 0$ and as a consequence the restriction of $g^{(\alpha^*)}$ to $X^{(\alpha^*)}$ is nonzero at the origin $x_1 = \dots = x_n = 0$. So the vanishing order of the restriction of $g^{(\alpha^*)}$ on $X^{(\alpha^*)}$ at the origin $x_1 = \dots = x_n = 0$ is just 0. So we can assume without loss of generality that $1 \leq p \leq m$.

By using a \mathbb{C} -linear transformation of the inhomogeneous coordinates x_1, \dots, x_n and, after dividing $g^{(\alpha)}$ by a local holomorphic function on \mathbb{P}_N and replacing W by another nonempty open subset of \mathbb{P}_N , we can assume without loss of generality that $g^{(\alpha)}$ is of the form

$$(4.1.1) \quad x_n^p + \sum_{j=0}^{m-p} A_j(\alpha, x_1, \dots, x_{n-1}) x_n^j,$$

where each $A_j(\alpha, x_1, \dots, x_{n-1})$ is a polynomial in x_1, \dots, x_{n-1} of degree $\leq m - j$ whose coefficients are holomorphic functions on W such that $A_j(\alpha, x_1, \dots, x_{n-1})$ vanishes at $x_1 = \dots = x_{n-1} = 0$ for $0 \leq j \leq p$.

The expression (4.1.1) is simply writing $g^{(\alpha)}$ as a polynomial in x_n , where a \mathbb{C} -linear transformation of x_1, \dots, x_n is used to make sure that the coefficient of x_n^p in the initial polynomial of degree p , at the origin in the inhomogeneous coordinates, is a non identically zero function of α , which is reduced to 1 after dividing $g^{(\alpha)}$ by it. The condition that $A_j(\alpha, x_1, \dots, x_{n-1})$ vanishes at $x_1 = \dots = x_{n-1} = 0$ for $0 \leq j < p$ simply means that the vanishing order at the origin of the inhomogeneous coordinates is indeed p and not less than p . The condition that $A_j(\alpha, x_1, \dots, x_{n-1})$ vanishes at $x_1 = \dots = x_{n-1} = 0$ for $j = p$ simply means that its constant term has been separated out to be the first term x_n^p after division by its coefficient which is a function of α .

We apply to the restriction of $g^{(\alpha)}$ to \mathcal{X} the tangent vector field

$$\xi = \frac{\partial}{\partial x_n} + \sum_{\nu_0 + \dots + \nu_n = \delta} B_{\nu_0, \dots, \nu_n}(\alpha) \frac{\partial}{\partial \alpha_{\nu_0, \dots, \nu_n}}$$

of \mathcal{X} (from the proof of Lemma 1.6) p times and get

$$\xi^p g^{(\alpha)} = p! + \sum_{j=0}^m \tilde{A}_j(\alpha, x_1, \dots, x_{n-1}) x_n^j,$$

where $\tilde{A}_0(\alpha, x_1, \dots, x_{n-1})$ vanishes at $x_1 = \dots = x_{n-1} = 0$. In particular, $\xi^p g^{(\alpha)}$ is nonzero at the origin of \mathbb{P}_n for any α .

Let $Y = \sum_{\ell} q_{\ell} Y_{\ell}$ be the divisor of the restriction of $g^{(\alpha)}$ to \mathcal{X} , where Y_{ℓ} is an irreducible hypersurface in $\mathcal{X} \cap (\mathbb{P}_n \times W)$. Since the p times differentiation $\xi^p g^{(\alpha)}$ of $g^{(\alpha)}$ with respect to the vector field ξ is nonzero at every point of $\mathcal{X} \cap (\{0\} \times W)$, it follows that the multiplicity of Y at every point of $\mathcal{X} \cap (\{0\} \times W)$ is at most p . If $\{0\} \times W$ is not contained in Y_{ℓ} , we let Y'_{ℓ} be $Y_{\ell} \cap (\{0\} \times W)$, otherwise we let Y'_{ℓ} be the empty set. Then for $\alpha \in W$ which is not in the image of the projection of $\bigcup_{\ell} Y'_{\ell}$ to W , the vanishing order of the restriction $g^{(\alpha)}|_{X(\alpha)}$ at the origin $x_1 = \dots = x_n = 0$ is $\leq p \leq m$.

Note that in this proof of the first statement we already fully use the special form of \mathcal{X} defined by

$$f^{(\alpha)} = \sum_{\nu_0 + \dots + \nu_n = \delta} \alpha_{\nu_0, \dots, \nu_n} x_1^{\nu_1} \dots x_n^{\nu_n}$$

in our use of the vector field

$$\xi = \frac{\partial}{\partial x_n} - \sum_{\substack{\nu_0 + \dots + \nu_n = \delta, \\ \nu_n \geq 1}} \nu_n \alpha_{\nu_0, \dots, \nu_n} \frac{\partial}{\partial \alpha_{\nu_0, \dots, \nu_{n-1}, \nu_n-1}},$$

which comes from the proof of Lemma 1.6 by setting $z_0 = 1, z_j = x_j$ for $1 \leq j \leq n$ and setting $a_{j,k} = 0$ for $(j,k) \neq (0,n)$ and $a_{0,n} = 1$. One can also easily get ξ directly from

$$\frac{\partial}{\partial x_n} f^{(\alpha)} = \sum_{\substack{\nu_0 + \dots + \nu_n = \delta, \\ \nu_n \geq 1}} n_n \alpha_{\nu_0, \dots, \nu_n} x_1^{\nu_1} \dots x_{n-1}^{\nu_{n-1}} x_n^{\nu_n-1}$$

and then from replacing the monomial $x_1^{\nu_1} \dots x_{n-1}^{\nu_{n-1}} x_n^{\nu_n-1}$ by

$$\frac{\partial}{\partial \alpha_{\nu_0, \dots, \nu_{n-1}, \nu_n-1}} f^{(\alpha)} = x_1^{\nu_1} \dots x_{n-1}^{\nu_{n-1}} x_n^{\nu_n-1}.$$

Now we come to the second statement without the local holomorphic section σ of \mathcal{X} over W . Let V be the set of all points (P, α) of $\mathbb{P}_n \times W$

such that the vanishing order at P of the restriction $g^{(\alpha)}|_{X^{(\alpha)}}$ of $g^{(\alpha)}$ to $X^{(\alpha)}$ is $> m$. Then V is a subvariety of $\mathcal{X} \cap (\mathbb{P}_n \times W)$. If the second statement does not hold, then the projection of V to W induced by $\mathbb{P}_n \times W \mapsto W$ is surjective. We can find a local holomorphic section σ of V over some nonempty open subset W' of W . This contradicts the first statement involving the local holomorphic section σ of \mathcal{X} over the nonempty open subset W' of \mathbb{P}_N . \square

Remark 4.2. (1) For use of the Lemma 4.1 in our proof of hyperbolicity of generic hypersurface of high degree, we need only a very weak version of it, for example, with the conclusion that the vanishing order of the restriction of $g^{(\alpha)}$ to $X^{(\alpha)}$ is $\leq c_n m^{1+\varepsilon_n} + c'_n$ at any point of its zero-set in $X^{(\alpha)}$ for some positive integers c_n and c'_n which depend only on n , where ε_n is some small positive number depending only on n .

(2) The generic condition on α in Lemma 4.1 cannot be removed. A simple example to illustrate this point is the Fermat hypersurface $\sum_{j=1}^n x_j^\delta = 1$ of degree δ in \mathbb{P}_n , for which the complex hyperplane $x_n = 1$ touches the Fermat hypersurface to order δ , because the equation $x_n = 1$ when pulled back to the Fermat hypersurface yields $\sum_{j=1}^{n-1} x_j^\delta$ which vanishes to order δ at the point $(x_1, \dots, x_{n-1}, x_n)$ with $x_j = 0$ for $1 \leq j \leq n-1$ and $x_n = 1$.

Proposition 4.3. (Jet Differentials Nonzero at Jets Represented by Nonsingular Curve Germ Obtained by Applying Slanted Vector Fields)

Let $\omega^{(\alpha)} = \frac{Q^{(\alpha)}}{f_{x_1}^{(\alpha)} - 1}$ be the holomorphic jet differential on $X^{(\alpha)}$ constructed in (3.8). There exists some proper subvariety \mathcal{Z} of \mathbb{P}_N such that for $a \in \mathbb{P}_N - \mathcal{Z}$ and any $P_0 \in J_{n-1}(X^{(\alpha)})$ representable by a nonsingular complex curve germ in $X^{(\alpha)}$, there exist slanted vector fields of low vertical pole order ξ_1, \dots, ξ_ℓ (with ℓ explicitly of order δ^ε) such that $\mathcal{L}ie_{\xi_1} \cdots \mathcal{L}ie_{\xi_\ell} \omega^{(\alpha)}$ defines a holomorphic jet differential on $X^{(\alpha)}$ which is nonzero at the $(n-1)$ -jet P_0 .

Proof. From our construction in (3.8) the k -jet differential $Q^{(\alpha)}$ is a polynomial of low degree in both the inhomogeneous coordinates x_1, \dots, x_n and their differentials $d^\nu x_1, \dots, d^\nu x_n$ (for $1 \leq \nu \leq n-1$) so that for a generic a the pullback to $X^{(\alpha)}$ of the k -jet differential $\omega^{(\alpha)} := \frac{Q^{(\alpha)}}{F^{(\alpha)}}$ is holomorphic on $X^{(\alpha)}$, where $F^{(\alpha)} = f_{x_1}^{(\alpha)} - 1$ is a polynomial in x_1, \dots, x_n of high degree (i.e., comparable to the degree δ of f) and dividing by it contributes to high vanishing order at the infinity hyperplane of \mathbb{P}_n . We introduce the new notation $F^{(\alpha)}$ here to emphasize that in the following argument the coordinate x_1 plays no role and what

matters is the contribution of division by $F^{(\alpha)}$ to high vanishing order at the infinity hyperplane of \mathbb{P}_n . We are interested in the vanishing order of the pullback to $X^{(\alpha)}$ of the k -jet differential $\frac{Q^{(\alpha)}}{F^{(\alpha)}}$ on $X^{(\alpha)}$ and how, if the hypersurface $X^{(\alpha)}$ is generic, at any given point of $X^{(\alpha)}$, an effectively small number of Lie differentiations can reduce it to no vanishing.

First of all, for notational simplicity we introduce the following notation for a monomial of differentials of inhomogeneous coordinates. Take monomial

$$\prod_{1 \leq j \leq n, 1 \leq \ell \leq k} (d^\ell x_j)^{\nu_{\ell,j}}$$

of differentials of the inhomogeneous coordinates x_1, \dots, x_n of \mathbb{P}_n up to order k , where each $\nu_{\ell,j}$ is a nonnegative integer. We use $\boldsymbol{\nu}$ to denote the indexed collection $(\nu_{\ell,j})_{1 \leq j \leq n, 1 \leq \ell \leq k}$ of nonnegative integers and use $d^{\boldsymbol{\nu}} \mathbf{x}$ to denote the monomial

$$\prod_{1 \leq j \leq n, 1 \leq \ell \leq k} (d^\ell x_j)^{\nu_{\ell,j}}$$

itself. To the monomial $d^{\boldsymbol{\nu}} \mathbf{x}$ of differentials of inhomogeneous coordinates up to order k define its multi-degree as the k -tuple of nonnegative integers $(m_k, m_{k-1}, \dots, m_1)$, where $m_\ell = \sum_{j=1}^n \nu_{\ell,j}$ is its degree as a monomial in the variables $d^\ell x_1, \dots, d^\ell x_n$ when the other differentials are regarded as coefficients. We use the notation $\text{mdg}(d^{\boldsymbol{\nu}} \mathbf{x})$ to denote the multi-degree $(m_k, m_{k-1}, \dots, m_1)$. Since the multi-degree of $d^{\boldsymbol{\nu}} \mathbf{x}$ depends only on the indexed collection $\boldsymbol{\nu}$ of nonnegative integers, we denote the multi-degree $(m_k, m_{k-1}, \dots, m_1)$ also simply by $\text{mdg}(\boldsymbol{\nu})$.

We now write

$$Q^{(\alpha)} = \sum_{\boldsymbol{\nu}} Q_{\boldsymbol{\nu}}^{(\alpha)} (d^{\boldsymbol{\nu}} \mathbf{x}),$$

where

$$Q_{\boldsymbol{\nu}}^{(\alpha)} = \sum_{\lambda_1, \dots, \lambda_n} Q_{\boldsymbol{\nu}, \ell_n; \lambda_1, \dots, \lambda_n}^{(\alpha)} x_1^{\lambda_1} \cdots x_n^{\lambda_n}.$$

We say that a k -tuple $(m_k, m_{k-1}, \dots, m_1)$ is represented as a multi-degree in $Q^{(\alpha)}$ if there exists some multi-index $\boldsymbol{\nu}$ such that $(m_k, m_{k-1}, \dots, m_1)$ is equal to $\text{mdg}(\boldsymbol{\nu})$ and the polynomial $Q_{\boldsymbol{\nu}}^{(\alpha)}$ of inhomogeneous coordinates x_1, \dots, x_n of \mathbb{P}_n is not identically zero.

Take an arbitrary point y of $X^{(\alpha)}$. For a multi-index $\boldsymbol{\nu}$ with $Q_{\boldsymbol{\nu}}^{(\alpha)}$ not identically zero as a polynomial of inhomogeneous coordinates x_1, \dots, x_n of \mathbb{P}_n , let $q_{y, \boldsymbol{\nu}}$ be the vanishing order at y of the restriction of $Q_{\boldsymbol{\nu}}^{(\alpha)}$ to $X^{(\alpha)}$. This vanishing order $q_{y, \boldsymbol{\nu}}$ is effectively bounded from

above by Lemma 4.1 for a generic α . For a k -tuple $(m_k, m_{k-1}, \dots, m_1)$ which is represented as multi-degree in $Q^{(\alpha)}$, let $q_{y, m_k, m_{k-1}, \dots, m_1}$ be the minimum of $q_{y, \nu}$ over all multi-indices ν with $\text{mdg}(\nu) = (m_k, m_{k-1}, \dots, m_1)$. Note that the number $q_{y, m_k, m_{k-1}, \dots, m_1}$ remains unchanged under any \mathbb{C} -linear change of the inhomogeneous coordinates x_1, \dots, x_n , even though $Q^{(\alpha)}$ as well as $q_{y, \nu}$ may be changed under a \mathbb{C} -linear change of the inhomogeneous coordinates x_1, \dots, x_n .

We now pull back $Q^{(\alpha)}$ to $X^{(\alpha)}$ to get $\iota^*Q^{(\alpha)}$ and express $\iota^*Q^{(\alpha)}$ in terms of the coordinates x_1, \dots, x_{n-1} and their differentials. We will show that the result of applying an effective number of well chosen slanted vector fields ξ_1, \dots, ξ_p to $\iota^*Q^{(\alpha)}$ will be a holomorphic jet differential on $X^{(\alpha)}$ which is pointwise non identically zero at y (for a generic $\alpha \in \mathbb{P}_n$ with the generic condition not depending on y).

In the set of all k -tuples of nonnegative integers $(m_k, m_{k-1}, \dots, m_1)$, we introduce the lexicographical ordering so that $(m_k, m_{k-1}, \dots, m_1)$ is greater than $(m'_k, m'_{k-1}, \dots, m'_1)$ if there exists some $1 \leq \hat{\lambda} \leq k$ such that $m_\lambda = m'_\lambda$ for $\lambda \geq \hat{\lambda}$ and $m_{\hat{\lambda}} > m'_{\hat{\lambda}}$. Let $(m_k^*, m_{k-1}^*, \dots, m_1^*)$ be the largest k -tuple among all k -tuples which are represented as multi-degrees in $Q^{(\alpha)}$. Let λ^* be the largest integer among all $1 \leq \lambda \leq k$ with $m_\lambda^* > 0$.

We are going to use an inhomogeneous coordinate system x_1, \dots, x_n of \mathbb{P}_n centered at y as follows. First of all, we choose x_n such that the hyperplane $\{x_n = 0\}$ is the tangent hyperplane of $X^{(\alpha)}$ at the point y . Secondly, we generically choose \mathbb{C} -linear functions x_1, \dots, x_{n-1} so that $q_{y, m_k^*, m_{k-1}^*, \dots, m_1^*} = q_{y, \nu^*}$, where ν^* is the indexed collection of nonnegative integers such that

$$d^{\nu^*} \mathbf{x} = (d^{\lambda^*} x_1)^{m_{\lambda^*}^*} (d^{\lambda^*-1} x_1)^{m_{\lambda^*-1}^*} \dots (d^2 x_1)^{m_2^*} (dx_1)^{m_1^*}.$$

In order to express $\iota^*Q^{(\alpha)}$ in terms of the coordinates x_1, \dots, x_{n-1} and their differentials, we use the implicit function theorem to express x_n in terms of x_1, \dots, x_{n-1} on $X^{(\alpha)}$ to get a local holomorphic function $x_n = g_{\alpha, u}(x_1, \dots, x_{n-1})$ in a neighborhood of y in \mathbb{P}_n . Since $\{x_n = 0\}$ is the tangent hyperplane of $X^{(\alpha)}$ at the point y , it follows that $\frac{\partial g_{\alpha, u}}{\partial x_1}$ vanishes at y .

For any λ^* -tuple of nonnegative integers $\mathbf{t} = (t_\lambda)_{1 \leq \lambda \leq \lambda^*}$ with $0 \leq t_\lambda \leq m_\lambda^*$ for $1 \leq \lambda \leq \lambda^*$ let $\nu_{\mathbf{t}}^*$ be the indexed collection of nonnegative

integers such that

$$\left(\prod_{\lambda=1}^{\lambda^*} (d^\lambda x_1)^{t_\lambda} \right) d^{\nu^*} \mathbf{x} = \left(\prod_{\lambda=1}^{\lambda^*} (d^\lambda x_n)^{t_\lambda} \right) d^{\nu^*} \mathbf{x},$$

which means that $d^{\nu^*} \mathbf{x}$ is obtained from $d^{\nu^*} \mathbf{x}$ by replacing the factor $(d^\lambda x_1)^{t_\lambda}$ in $d^{\nu^*} \mathbf{x}$ by the factor $(d^\lambda x_n)^{t_\lambda}$ for $1 \leq \lambda \leq \lambda^*$.

The coefficient of $d^{\nu^*} \mathbf{x}$ in $Q^{(\alpha)}$ is $Q_{\nu^*}^{(\alpha)}$, whereas the coefficient of $d^{\nu^*} \mathbf{x}$ in $\iota^* Q^{(\alpha)}$ is given by

$$Q_{\nu^*}^{(\alpha)} + \sum_{\substack{0 \leq t_\lambda \leq m_\lambda^* \text{ for } 1 \leq \lambda \leq \lambda^* \\ \text{and } \sum_{\lambda=1}^{\lambda^*} (m_\lambda^* - t_\lambda) > 0}} \left(\frac{\partial g_{\alpha, u}}{\partial x_1} \right)^{\sum_{\lambda=1}^{\lambda^*} (m_\lambda^* - t_\lambda)} Q_{\nu_t^*}^{(\alpha)}.$$

The vanishing order at y of the restriction $\iota^* Q_{\nu^*}^{(\alpha)}$ of $Q_{\nu^*}^{(\alpha)}$ to $X^{(\alpha)}$ is equal to $q_{y, m_k^*, m_{k-1}^*, \dots, m_1^*}$, whereas the y of the restriction $\iota^* Q_{\nu_t^*}^{(\alpha)}$ of $Q_{\nu_t^*}^{(\alpha)}$ to $X^{(\alpha)}$ no more than $q_{y, m_k^*, m_{k-1}^*, \dots, m_1^*}$ when $0 \leq t_\lambda \leq m_\lambda^*$ for $1 \leq \lambda \leq \lambda^*$ and $\sum_{\lambda=1}^{\lambda^*} (m_\lambda^* - t_\lambda) > 0$.

Since the vanishing order at y of $\iota^* Q_{\nu^*}^{(\alpha)}$ is equal to $q_{y, m_k^*, m_{k-1}^*, \dots, m_1^*}$, by Proposition (2.17) on the global generation on jet space by slanted vector fields at points representable by regular curve germs, we can choose slanted vector fields ξ_j for $1 \leq j \leq q$ of low vertical pole order with $q = q_{y, m_k^*, m_{k-1}^*, \dots, m_1^*}$ such that $\xi_1 \cdots \xi_q (\iota^* Q_{\nu^*}^{(\alpha)})$ is nonzero at y and $\mathcal{L}ie_{\xi_\lambda} \mathcal{L}ie_{\xi_{\lambda+1}} \cdots \mathcal{L}ie_{\xi_q} (d^\ell x_j)$ on \mathcal{X} equals $d^\ell x_j$ at y for $1 \leq \lambda \leq q$, $1 \leq \ell \leq k$ and $1 \leq j \leq n-1$, because slanted vector fields generate the tangent bundle of the space of vertical jet differentials of \mathcal{X} . To choose such slanted vector fields of low vertical pole order, we need only make sure that their components along the local coordinates x_1, \dots, x_{n-1} of \mathcal{X} (which are now regarded as local coordinates of the k -jet space of \mathcal{X} by pullback) give a nonzero value of $\xi_1 \cdots \xi_q (\iota^* Q_{\nu^*}^{(\alpha)})$ at y and their components along the coordinates $d^\ell x_j$ for $1 \leq \ell \leq k$ and $1 \leq j \leq n-1$ in the k -jet space of \mathcal{X} are all zero at y . Since the vanishing order at y of $\iota^* Q_{\nu^*}^{(\alpha)}$ is equal to q , we have the vanishing of $\xi_\lambda \xi_{\lambda+1} \cdots \xi_q (\iota^* Q_{\nu^*}^{(\alpha)})$ for $2 \leq \lambda \leq q$.

Since the vanishing order at y of $\iota^* Q_{\nu_t^*}^{(\alpha)}$ is no more than q and since $\frac{\partial g_{\alpha, y}}{\partial x_1}$ vanishes at y , we have the vanishing of

$$\xi_\lambda \xi_{\lambda+1} \cdots \xi_q \left(\left(\frac{\partial g_{\alpha, y}}{\partial x_1} \right)^{\sum_{\lambda=1}^{\lambda^*} (m_\lambda^* - t_\lambda)} Q_{\nu_t^*}^{(\alpha)} \right)$$

at y for $1 \leq \lambda \leq q$ when $0 \leq t_\lambda \leq m_\lambda^*$ for $1 \leq \lambda \leq \lambda^*$ and $\sum_{\lambda=1}^{\lambda^*} (m_\lambda^* - t_\lambda) > 0$.

We now apply the Lie differentiation by ξ_1, \dots, ξ_q to $\omega^{(\alpha)} = \frac{Q^{(\alpha)}}{F^{(\alpha)}}$ and conclude that at y the coefficient of $d^{\nu^*} \mathbf{x}$ in $\mathcal{L}ie_{\xi_1} \mathcal{L}ie_{\xi_2} \cdots \mathcal{L}ie_{\xi_q} (\omega^{(\alpha)})$ is equal to the value of $\xi_1 \cdots \xi_q (\iota^* Q_{\nu^*}^{(\alpha)})$ at y which is nonzero.

Take any P_0 of $J_{n-1}^{\text{vert}}(\mathcal{X})$ above the point y of $X^{(\alpha)}$ which is representable by a nonsingular complex curve germ in $X^{(\alpha)}$. As in (2.21), we can choose slanted vector fields $\eta_1, \dots, \eta_{\hat{m}}$ of low vertical pole order (from Proposition 2.19) with \hat{m} not exceeding the weight m of $\mathcal{L}ie_{\xi_1} \cdots \mathcal{L}ie_{\xi_q} (\omega^{(\alpha)})$ such that the holomorphic $(n-1)$ -jet differential $\mathcal{L}ie_{\eta_1} \cdots \mathcal{L}ie_{\eta_{\hat{m}}} \mathcal{L}ie_{\xi_1} \cdots \mathcal{L}ie_{\xi_q} (\omega^{(\alpha)})$ on $X^{(\alpha)}$ vanishing on an ample divisor of $X^{(\alpha)}$ has a nonzero value at P_0 . \square

4.4. Proof of Theorem 0.1

For a generic hypersurface $X^{(\alpha)}$ of sufficiently high degree δ , by Proposition 4.3 for every P_0 in the $(n-1)$ -jet space $J_{n-1}(X^{(\alpha)})$ representable by nonsingular complex curve germ there exists a holomorphic $(n-1)$ -jet differential ω on $X^{(\alpha)}$ which vanishes on an ample divisor of $X^{(\alpha)}$ and which has nonzero value at P_0 .

We need to verify that there exists no nonconstant holomorphic map $\varphi : \mathbb{C} \rightarrow X^{(\alpha)}$. Suppose the contrary and we are going to derive a contradiction. Since φ is nonconstant, there exists some $\zeta_0 \in \mathbb{C}$ with the differential $d\varphi$ of φ is nonzero. Let P_0 be the $(n-1)$ -jet of $X^{(\alpha)}$ at $\varphi(\zeta_0)$ represented by the parametrized complex curve germ $\varphi : \mathbb{C} \rightarrow X^{(\alpha)}$. Since ω is a holomorphic $(n-1)$ -jet differential on $X^{(\alpha)}$ vanishing on an ample divisor of $X^{(\alpha)}$, it follows that the pullback $\varphi^* \omega$ is identically zero on \mathbb{C} , contradicting the value of ω at P_0 being nonzero (*c.f.*, Proposition 5.4 and Remark 5.5 below).

4.5. Proof of Theorem (0.2)

The proof is analogous to the proof of Theorem 0.1. The difference is that

- (i) the holomorphic jet differential ω used in the proof of Theorem 0.1 is replaced by the log-pole jet differential on \mathbb{P}_n constructed in Theorem 5.13,
- (ii) the use of Lemma 4.1 in the proof of Proposition 4.3 is modified by Remark 5.14 concerning the vanishing order of log-pole jet differential constructed as the direct image of jet differential,

- (iii) the use of Proposition 2.17 is replaced by the use of Proposition 2.19
- (iv) the use of the Schwarz lemma of the vanishing of the pullback to \mathbb{C} of a holomorphic jet differential vanishing on an ample divisor is replaced by Proposition 5.10 concerning the general Schwarz lemma for log-pole jet differentials.

5. ESSENTIAL SINGULARITIES, VARYING COEFFICIENTS, AND SECOND MAIN THEOREM

Besides the Little Picard Theorem of nonexistence of nonconstant holomorphic maps from \mathbb{C} to $\mathbb{P}_1 - \{0, 1, \infty\}$ there is a stronger statement which is the Big Picard Theorem of no essential singularity at ∞ for any holomorphic function from $\mathbb{C} - \overline{\Delta_{r_0}}$ to $\mathbb{C} - \{0, 1\}$ for $r_0 > 0$. The Little Picard Theorem corresponds to our theorem on the hyperbolicity of a generic hypersurface $X^{(\alpha)}$ of sufficiently high degree in \mathbb{P}_n (for α in \mathbb{P}_N outside a proper subvariety \mathcal{Z} of \mathbb{P}_N). Corresponding to the Big Picard Theorem, there is a statement concerning the extendibility across a holomorphic map $\mathbb{C} - \overline{\Delta_{r_0}} \rightarrow X^{(\alpha)}$ to a holomorphic map $\mathbb{C} \cup \{\infty\} - \overline{\Delta_{r_0}} \rightarrow X^{(\alpha)}$.

In this section we are going to prove such a theorem on removing the essential singularity at ∞ of a holomorphic map from $\mathbb{C} - \overline{\Delta_{r_0}}$ to a generic hypersurface of sufficiently high degree.

The more quantitative version of the Big Picard Theorem was introduced by Nevanlinna [Ne25] in his Second Main Theorem in his theory of value distribution theory. In this section we discuss the Second Main Theorem from log-pole jet differentials which is more in the context of Cartan's generalization of Nevanlinna's Second Main Theorem to holomorphic maps from the affine complex line to \mathbb{P}_n and a collection of hyperplanes in general position [Ca33].

The hyperbolicity of a generic hypersurface of high degree δ in \mathbb{P}_n can be reformulated as the nonexistence of $n + 1$ entire holomorphic functions on \mathbb{P} with some ratio nonconstant which satisfy a generic homogeneous polynomial of degree δ with constant coefficients. Our solution of the hyperbolicity problem for a generic hypersurface of high degree makes use of the universal hypersurface \mathcal{X} in $\mathbb{P}_n \times \mathbb{P}_N$ and the variation of the hypersurface $X^{(\alpha)}$ in \mathbb{P}_n with $\alpha \in \mathbb{P}_N$. The variation of $X^{(\alpha)}$ corresponds to the varying of the constant coefficients of the homogeneous polynomial equation for the $n + 1$ entire functions. In this section we will discuss the problem of nonexistence of entire functions satisfying polynomial equations with slowly varying coefficients

and also the more general result for removing essential singularities for holomorphic functions on \mathbb{C} minus a disk for this setting.

I. REMOVAL OF ESSENTIAL SINGULARITIES

To extend our methods from maps $\mathbb{C} \rightarrow X^{(\alpha)}$ to maps $\mathbb{C} - \overline{\Delta_{r_0}} \rightarrow X^{(\alpha)}$ for some $r_0 > 0$, we need a corresponding extension of Nevanlinna's logarithmic derivative lemma (p.51 of [Ne25]). For such an extension of Nevanlinna's logarithmic derivative lemma, we need the following trivial multiplicative version of the Heftungslemma [AN64].

Lemma 5.1. (Trivial Multiplicative Version of Heftungslemma) *Let $r_0 > 0$ and F be meromorphic on $\mathbb{C} - \overline{\Delta_{r_0}}$. Let $r_0 < r_1$. Then there exists some function G holomorphic and nowhere zero on $\mathbb{C} \cup \{\infty\} - \overline{\Delta_{r_1}}$ such that FG is meromorphic on \mathbb{C} . Moreover, when F is holomorphic, G can be chosen so that FG is also holomorphic on $\mathbb{C} - \{0\}$.*

Proof. Choose $r_0 < \rho_1 < r_1 < \rho_2$ such that $\partial\Delta_{\rho_j}$ contains no pole and no zero of F for $j = 1, 2$. Let $\sum_{j=1}^{J_0} a_j$ be the zero-divisor of F on $\overline{\Delta_{\rho_2}} - \Delta_{\rho_1}$ and $\sum_{j=J_\infty} b_j$ be the pole-divisor of F on $\overline{\Delta_{\rho_2}} - \Delta_{\rho_1}$. Let

$$h = \left(\prod_{j=J_0} (\zeta - a_j) \right)^{-1} \left(\prod_{j=J_\infty} (\zeta - b_j) \right).$$

Then Fh is holomorphic and nowhere zero on $\overline{\Delta_{\rho_2}} - \Delta_{\rho_1}$. Let ℓ_0 be the integer

$$\frac{1}{2\pi} \int_{|\zeta|=r_1} d \log(Fh).$$

Then

$$\int_{|\zeta|=r_1} d \log(Fh\zeta^{-\ell_0}) = 0$$

and we can define a branch of $\log(Fh\zeta^{-\ell_0})$ on $\overline{\Delta_{\rho_2}} - \Delta_{\rho_1}$, which we denote by Φ . From Cauchy's integral formula $\Phi(\zeta) = \Phi_0(\zeta) - \Phi_\infty$, where

$$\begin{aligned} \Phi_0(\zeta) &= \frac{1}{2\pi} \int_{|\hat{\zeta}|=\rho_2} \frac{\Phi(\hat{\zeta})}{\hat{\zeta} - \zeta} d\hat{\zeta}, \\ \Phi_\infty(\zeta) &= \frac{1}{2\pi} \int_{|\hat{\zeta}|=\rho_2} \frac{\Phi(\hat{\zeta})}{\hat{\zeta} - \zeta} d\hat{\zeta} \end{aligned}$$

for $\zeta \in \Delta_{\rho_2} - \overline{\Delta_{\rho_1}}$. Exponentiating both sides of $\Phi(\zeta) = \Phi_0(\zeta) - \Phi_\infty$, we get $Fh\zeta^{-\ell_0} = e^{\Phi_0} e^{-\Phi_\infty}$ and $F e^{\Phi_\infty} = h^{-1} \zeta^{\ell_0} e^{\Phi_0}$. Since the right-hand side $h^{-1} \zeta^{\ell_0} e^{\Phi_0}$ of $F e^{\Phi_\infty} = h^{-1} \zeta^{\ell_0} e^{\Phi_0}$ is meromorphic on Δ_{ρ_2} and

the left-hand side of $Fe^{\Phi_\infty} = h^{-1}\zeta^{\ell_0}e^{\Phi_0}$ is meromorphic on $\mathbb{C} - \overline{\Delta_{\rho_1}}$, it follows Fe^{Φ_∞} is meromorphic on all of \mathbb{C} . We apply the transformation $w = \frac{1}{\zeta}$ to get

$$\Phi_\infty = \frac{1}{2\pi} \int_{|\hat{\zeta}|=\rho_1} \frac{\Phi(\hat{\zeta})}{\hat{\zeta} - \zeta} d\hat{\zeta} = \frac{1}{2\pi} \int_{|\hat{\zeta}|=\rho_1} \frac{w\Phi(\zeta)}{w\zeta - 1} d\hat{\zeta}$$

which is holomorphic for $|w| < \frac{1}{\rho_1}$, that is, holomorphic for $\zeta_0 \in \mathbb{C} \cup \{\infty\} - \overline{\Delta_{\rho_1}}$. Now the function $G(\zeta) = e^{\Phi_\infty}$ satisfies our requirement. When F is holomorphic on $\mathbb{C} - \overline{\Delta_{r_0}}$, the function FG which is equal to $h^{-1}\zeta^{\ell_0}e^{\Phi_0}$ on Δ_{ρ_2} and is equal to e^{Φ_0} on $\mathbb{C} - \overline{\Delta_{\rho_1}}$ is clearly holomorphic on $\mathbb{C} - \{0\}$. \square

5.2. Comparison of Characteristic Functions of Maps Defined Outside a Disk.

Let $r_1 > r_0 > 0$. For a meromorphic function H on $\mathbb{C} - \overline{\Delta_{r_0}}$ we introduce for $c \in \mathbb{C} \cup \{\infty\}$ the counting function

$$N(r, r_1, H, c) = \int_{\rho=r_1}^r n(\rho, r_1, H, c) \frac{d\rho}{\rho},$$

where $n(\rho, r_1, H, c)$ is the number of roots of $H(\zeta) = c$ with multiplicities counted in $r_1 \leq |\zeta| \leq \rho$ and also the characteristic function

$$T(r, r_1, H) = N(r, r_1, H, \infty) + \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \log^+ |H(re^{i\theta})| d\theta.$$

For a holomorphic map φ from $\mathbb{C} - \overline{\Delta_{r_0}}$ to a complex manifold Y and a $(1, 1)$ -form η on Y , we introduce the characteristic function

$$T(r, r_1, \varphi, \eta) = \int_{\rho=r_1}^r \left(\int_{\Delta_r - \overline{\Delta_{r_1}}} \varphi^* \eta \right) \frac{d\rho}{\rho}.$$

For $Y = \mathbb{P}_n$ and η being the Fubini-Study form, we drop η in the notation $T(r, r_1, \varphi, \eta)$ and simply use $T(r, r_1, \varphi)$ when there is no confusion. When a holomorphic map φ from $\mathbb{C} - \overline{\Delta_{r_0}}$ to \mathbb{P}_n is given by holomorphic functions $[F_0, \dots, F_n]$ on $\mathbb{C} - \overline{\Delta_{r_0}}$ without common zeroes, its characteristic function is

$$T(r, r_1, \varphi) = \int_{\rho=r_1}^r \left(\int_{|\zeta|<\rho} \frac{1}{\pi} \partial_\zeta \partial_{\bar{\zeta}} \log \sum_{k=0}^N |F_k|^2 \right) \frac{d\rho}{\rho}.$$

We would like to compare it with the characteristic function $T\left(r, r_1, \frac{F_j}{F_0}\right)$ for the meromorphic function $\frac{F_j}{F_0}$ for $1 \leq j \leq n$. For $1 \leq j \leq n$ we have

the inequality

$$T\left(r, \frac{F_j}{F_0}\right) \leq T(r, \varphi) + O(1) \leq \sum_{k=1}^n T\left(r, \frac{F_k}{F_0}\right) + O(1).$$

The verification of

$$T\left(r, \frac{F_j}{F_0}\right) \leq T(r, \varphi) + O(1)$$

for $1 \leq j \leq n$ is as follows.

From twice integration of Laplacian with $g = \log \sum_{k=0}^N |F_k|^2$ in (0.12), we have

$$\begin{aligned} & \int_{\rho=r_1}^r \left(\int_{|\zeta|<\rho} \frac{1}{\pi} \partial_\zeta \partial_{\bar{\zeta}} \log \sum_{k=0}^N |F_k|^2 \right) \frac{d\rho}{\rho} \\ &= \frac{1}{4\pi} \int_{\theta=0}^{2\pi} \log \sum_{k=0}^N |F_k(r e^{i\theta})|^2 d\theta - \frac{1}{4\pi} \int_{\theta=0}^{2\pi} \log \sum_{k=0}^N |F_k(r_1 e^{i\theta})|^2 d\theta \end{aligned}$$

from which we conclude that

$$\begin{aligned} & \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \log^+ \left| \frac{F_j}{F_0}(r e^{i\theta}) \right| d\theta \\ & \leq \frac{1}{4\pi} \left(\int_{\theta=0}^{2\pi} \log \sum_{k=0}^N |F_k|^2 \right) \frac{d\rho}{\rho} \\ &= \int_{\rho=r_1}^r \left(\int_{|\zeta|<\rho} \frac{1}{\pi} \partial_\zeta \partial_{\bar{\zeta}} \log \sum_{k=0}^N |F_k|^2 \right) \frac{d\rho}{\rho} + \frac{1}{4\pi} \int_{\theta=0}^{2\pi} \log \sum_{k=0}^N |F_k(r_1 e^{i\theta})|^2 d\theta \\ &= T(r, \varphi) + O(\log r). \end{aligned}$$

Finally from

$$\begin{aligned} N(r, r_1, F_0, 0) &= \int_{\rho=r_1}^r \left(\frac{1}{\pi} \partial_\zeta \partial_{\bar{\zeta}} \log |F_0|^2 \right) \frac{d\rho}{\rho} \\ &= \int_{\rho=r_1}^r \left(\frac{1}{\pi} \partial_\zeta \partial_{\bar{\zeta}} \log \left(\frac{|F_0|^2}{\sum_{j=0}^N |F_0|^2} \cdot \sum_{j=0}^N |F_0|^2 \right) \right) \frac{d\rho}{\rho} \\ &\leq \int_{\rho=r_1}^r \left(\frac{1}{\pi} \partial_\zeta \partial_{\bar{\zeta}} \log \sum_{j=0}^N |F_0|^2 \right) \frac{d\rho}{\rho} = T(r, \varphi) \end{aligned}$$

it follows that

$$T\left(r, r_1, \frac{F_j}{F_0}\right) = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \log^+ \left| \frac{F_j}{F_0}(r e^{i\theta}) \right| d\theta + N(r, r_1, F_0, 0)$$

$$\leq O(T(r, r_1, \varphi) + \log r) \quad \parallel$$

The verification of

$$T(r, r_1, \varphi) \leq \sum_{j=k}^n T\left(r, r_1, \frac{F_k}{F_0}\right) + O(1)$$

for $1 \leq j \leq n$ is as follows. From

$$\begin{aligned} & \int_{\rho=r_1}^r \left(\int_{|\zeta|<\rho} \frac{1}{\pi} \partial_{\zeta} \partial_{\bar{\zeta}} \log \sum_{k=0}^N |F_k|^2 \right) \frac{d\rho}{\rho} \\ &= \frac{1}{4\pi} \int_{\theta=0}^{2\pi} \log \sum_{k=0}^N |F_k(re^{i\theta})|^2 d\theta - \frac{1}{4\pi} \int_{\theta=0}^{2\pi} \log \sum_{k=0}^N |F_k(r_1e^{i\theta})|^2 d\theta \\ &= \frac{1}{4\pi} \int_{\theta=0}^{2\pi} \log \left(1 + \sum_{k=1}^N \left| \frac{F_k}{F_0}(re^{i\theta}) \right|^2 \right) d\theta + \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \log |F_0(re^{i\theta})| d\theta \\ & \quad - \frac{1}{4\pi} \int_{\theta=0}^{2\pi} \log \sum_{k=0}^N |F_k(r_1e^{i\theta})|^2 d\theta \end{aligned}$$

we have

$$T(r, r_1, \varphi) \leq \sum_{k=1}^n \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \log^+ \left| \frac{F_k}{F_0}(re^{i\theta}) \right| d\theta + \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \log |F_0(re^{i\theta})| d\theta + O(1).$$

The verification of the inequality comparing characteristic functions of maps and meromorphic functions is a straightforward modification of the proof of Lemma (2.1.2) on p.426 of [Si95], where the map is from \mathbb{C} instead of from $\mathbb{C} - \overline{\Delta_{r_0}}$.

The logarithmic derivative lemma holds for meromorphic functions on $\mathbb{C} - \overline{\Delta_{r_0}}$ for $r_0 > 0$ in the following form.

Proposition 5.3. (Logarithmic Derivative Lemma for Functions Meromorphic in Punctured Disk Centered at Infinity) *Let $r_1 > r_0 > 0$ and F be meromorphic on $\mathbb{C} - \overline{\Delta_{r_0}}$. Then*

$$\int_{\theta=0}^{2\pi} \log^+ \left| \frac{F'(re^{i\theta})}{F(re^{i\theta})} \right| d\theta = O(\log T(r, r_1, F) + \log r) \quad \parallel$$

for $r > r_1$.

Proof. By 5.1 there exists some function G holomorphic and nowhere zero on $\mathbb{C} \cup \{\infty\} - \overline{\Delta_{r_1}}$ such that FG is meromorphic on \mathbb{C} . Let

$H = FG$. Then $(\log F)' = (\log H)' - (\log G)'$ and

$$\log^+ \left| \frac{F'(re^{i\theta})}{F(re^{i\theta})} \right| \leq \log^+ \left| \frac{H'(re^{i\theta})}{H(re^{i\theta})} \right| + \log^+ \left| \frac{G'(re^{i\theta})}{G(re^{i\theta})} \right| + \log 2.$$

Thus,

$$\begin{aligned} \int_{\theta=0}^{2\pi} \log^+ \left| \frac{F'(re^{i\theta})}{F(re^{i\theta})} \right| d\theta &\leq \int_{\theta=0}^{2\pi} \log^+ \left| \frac{H'(re^{i\theta})}{H(re^{i\theta})} \right| d\theta + \frac{1}{r} \int_{|\zeta|=r} \log^+ |dG| + \log 2 \\ &\leq \int_{\theta=0}^{2\pi} \log^+ \left| \frac{H'(re^{i\theta})}{H(re^{i\theta})} \right| d\theta + O(1) \leq O(\log T(r, H) + \log r) \quad \parallel \end{aligned}$$

because G holomorphic and nowhere zero on $\mathbb{C} \cup \{\infty\} - \overline{\Delta_{r_1}}$. The required statement follows from

$$N(r, H, \infty) \leq N(r, r_1, F, \infty) + O(\log r)$$

and $\log^+ |H| \leq \log^+ |F| + O(1)$. By 5.1 on the trivial multiplicative version of Heftungslemma, for some holomorphic nowhere-zero function G_0 on $\mathbb{C} \cup \{\infty\} - \overline{\Delta_{r_2}}$ with $r_0 < r_2 < r_1$ such that $H(\zeta) := \zeta^\ell F_0(\zeta) G_0(\zeta)$ is holomorphic on \mathbb{C} (with coordinate ζ) and is nonzero at $\zeta = 0$.

$$\begin{aligned} &\frac{1}{2\pi} \int_{\theta=0}^{2\pi} \log |F_0(re^{i\theta})| d\theta \\ &= \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \log \left| (re^{i\theta})^\ell (F_0 G_0(re^{i\theta})) \right| d\theta - \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \log \left| (re^{i\theta})^{-\ell} (G_0(re^{i\theta})) \right| d\theta \\ &= \log |H(0)| + N(r, H, 0) - \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \log \left| (re^{i\theta})^{-\ell} (G_0(re^{i\theta})) \right| d\theta \\ &= N(r, H, 0) + O(1) \leq N(r, r_1, F_0, 0) + O(1) \leq \sum_{k=1}^n N\left(r, r_1, \frac{F_k}{F_0}, \infty\right) + O(1) \end{aligned}$$

because F_0, F_1, \dots, F_n are assumed to have no common zeroes on $\mathbb{C} - \overline{\Delta_{r_0}}$. Thus,

$$\begin{aligned} T(r, \varphi) &\leq \sum_{k=1}^n \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \log^+ \left| \frac{F_k}{F_0}(re^{i\theta}) \right| d\theta + \sum_{k=1}^n N\left(r, r_1, \frac{F_k}{F_0}, \infty\right) + O(1) \\ &= \sum_{k=1}^n T\left(r, r_1, \frac{F_k}{F_0}\right) + O(1). \end{aligned}$$

□

Proposition 5.4. (Vanishing of Pullback of Jet Differential Vanishing on an Ample Divisor by Holomorphic Map to Punctured Disk Centered at Infinity) *Let X be a compact complex manifold with a Kähler form η*

and ω be a holomorphic jet differential on X vanishing on some ample divisor D of X . Let $r_1 > r_0 > 0$ and $\varphi : \mathbb{C} - \overline{\Delta_{r_0}} \rightarrow X$ be a holomorphic map. Let $\text{eval}_{\text{id}_{\mathbb{C}}}(\varphi^*\omega)$ denote the function on $\mathbb{C} - \overline{\Delta_{r_0}}$ whose value at ζ is the evaluation of the jet differential $\varphi^*\omega$ at the jet defined by the identity map of $\mathbb{C} - \overline{\Delta_{r_0}}$ at ζ . Then either $\text{eval}_{\text{id}_{\mathbb{C}}}(\varphi^*\omega)$ is identically zero on $\mathbb{C} - \overline{\Delta_{r_0}}$ or $T(r, r_1, \varphi, \eta) = O(\log r)$ \parallel .

Proof. Let k be the order of jet differential ω and m be its weight. Let L_D be the line bundle associated to the ample divisor D . Let $e^{-\chi_D}$ be a smooth metric of L_D whose curvature form η_D is strictly positive definite on X . Let s_D be a holomorphic section of L_D whose divisor is D . Let $\Phi = \text{eval}_{\text{id}_{\mathbb{C}}}(\varphi^*\omega)$. We assume that Φ is not identically zero. We apply twice integration of Laplacian in (0.12) to

$$g(\zeta) = \log \left(\frac{|\Phi|^2}{|s_D|^2 e^{-\chi_D}} \right).$$

Since ω is holomorphic on X and vanishing on D , it follows that

$$T(r, r_1, \varphi, \eta_D) \leq \frac{1}{4\pi} \int_{\theta=0}^{2\pi} \log \left(\frac{|\Phi|^2}{|s_D|^2 e^{-\chi_D}} \right) d\theta + O(1).$$

Here we have the inequality instead of an identity, because of possible contribution from the zero-set of $\frac{\omega}{s_D}$. At this point enters Bloch's technique of applying the logarithmic derivative lemma by using the logarithm of global meromorphic functions as local coordinates. As functions on the k -jet space $J_k(X)$ of X (with the right-hand side being global functions and the left-hand side being only local functions due to the transition functions of the line bundles L_D),

$$\left| \frac{\omega}{s_D} \right| \leq C \sum_{\lambda=1}^{\Lambda} \prod_{\nu_{j,\ell}, j,\ell} \left| d^\ell \log F_{j,\ell}^{(\lambda)} \right|^{\nu_{j,\ell}}$$

for some $C > 0$ and a finite collection $\{F_{j,\ell}^{(\lambda)}\}$ of global meromorphic functions on X , where the product is taken over the indices $\nu_{j,\ell}, j, \ell$ with the ranges $1 \leq j \leq n$, $1 \leq \ell \leq k$ and $\sum_{1 \leq j \leq n, 1 \leq \ell \leq k} \ell \nu_{j,\ell} = m$. By Nevanlinna's logarithmic derivative lemma (extended to \mathbb{C} outside a disk centered at the origin) given in (5.3),

$$\int_{\theta=0}^{2\pi} \log^+ \left| d^\ell \log F_{j,\ell}^{(\lambda)} \right| (r e^{i\theta}) d\theta = O(\log T(r, r_1, \varphi, \eta_D) + \log r) \parallel.$$

Hence

$$\int_{\theta=0}^{2\pi} \log \left(\frac{|\Phi|^2}{|s_D|^2 e^{-\chi_D}} \right) (r e^{i\theta}) d\theta = O(\log T(r, r_1, \varphi, \eta_D) + \log r) \quad \parallel$$

and we get

$$T(r, r_1, \varphi, \eta_D) \leq O(\log T(r, r_1, \varphi, \eta_D) + \log r) \quad \parallel$$

from which it follows that

$$T(r, r_1, \varphi, \eta_D) = O(\log r) \quad \parallel$$

and

$$T(r, r_1, \varphi, \eta) = O(\log r) \quad \parallel.$$

□

Remark 5.5. The argument in Proposition (5.4) is simply a modification of the case of the usual Schwarz lemma on pullbacks of jet differentials (see *e.g.*, Theorem 2 on p.1140 of [SY97]) when the holomorphic map φ is from the entire affine complex line \mathbb{C} to X . For this case, the pullback $\varphi^*\omega$ is always identically zero on \mathbb{C} for the following reason. We can replace φ by the composite ψ of φ with the exponential map from \mathbb{C} to \mathcal{C} to rule out the case of $T(r, \psi, \eta) = O(\log r) \parallel$ so that $\text{eval}_{\text{id}_{\mathbb{C}}}(\psi^*\omega)$ vanishes identically on \mathbb{C} . Since any k -jet of \mathbb{C} at any point ζ_0 of \mathbb{C} can be realized by some holomorphic map σ from \mathbb{C} to itself, from the vanishing of $\text{eval}_{\text{id}_{\mathbb{C}}}(\sigma^*\omega)$ vanishes identically on \mathbb{C} it follows that $\psi^*\omega$ is identically zero on \mathbb{C} , which implies that $\varphi^*\omega$ is identically zero on \mathbb{C} .

Lemma 5.6. (Extension of Holomorphic Maps with Log Order Growth Characteristic Function Across Infinity Point) *Let φ be a holomorphic map from $\mathbb{C} - \overline{\Delta_{r_0}}$ to \mathbb{P}_n given by holomorphic functions $[F_0, \dots, F_n]$ on $\mathbb{C} - \overline{\Delta_{r_0}}$ without common zeroes. If $T(r, r_1, \varphi) = O(\log r) \parallel$, then φ can be extended to a holomorphic map from $\mathbb{C} \cup \{\infty\} - \overline{\Delta_{r_0}}$ to \mathbb{P}_n .*

Proof. By the comparison of characteristic functions of maps defined outside a disk given in (5.2), we have

$$T\left(r, \frac{F_j}{F_0}\right) \leq T(r, \varphi) = O(\log r) \quad \parallel$$

for $1 \leq j \leq n$. By the trivial multiplicative version of the Hefungslemma given in (5.1), there exists some holomorphic nowhere zero function G_j on $\mathbb{C} \cup \{\infty\} - \overline{\Delta_{r_1}}$ such that $G_j \frac{F_j}{F_0}$ is meromorphic on \mathbb{C} . From

$$T\left(r, G_j \frac{F_j}{F_0}\right) = T\left(r, \frac{F_j}{F_0}\right) + O(1) \leq T(r, \varphi) = O(\log r) \quad \parallel$$

for $1 \leq j \leq n$ we conclude that $G_j \frac{F_j}{F_0}$ is a rational function on \mathbb{C} . Hence $\frac{F_j}{F_0}$ can be extended to a meromorphic function on $\mathbb{C} \cup \{\infty\} - \overline{\Delta_{r_0}}$. Thus φ can be extended to a holomorphic map from $\mathbb{C} \cup \{\infty\} - \overline{\Delta_{r_0}}$ to \mathbb{P}_n . \square

Proposition 5.7. *Let X be a compact complex manifold. Let $\omega_1, \dots, \omega_N$ be holomorphic k -jet differentials of total weight m on X with each vanishing on some ample divisor of X . Assume that at any point P_0 of $J_k(X)$ which is representable by a nonsingular complex curve germ, at least one of $\omega_1, \dots, \omega_N$ is nonzero at P_0 for some $1 \leq j \leq N$. Then any holomorphic map φ from $\mathbb{C} - \overline{\Delta_{r_0}}$ to X can be extended to a holomorphic map from $\mathbb{C} \cup \{\infty\} - \overline{\Delta_{r_0}}$ to X .*

Proof. We can assume without loss of generality that φ is not a constant map so that at some point ζ_0 of $\mathbb{C} - \overline{\Delta_{r_0}}$ the differential of φ is nonzero at ζ_0 . Let P_0 be the element of $J_k(X)$ at the point $\varphi(\zeta_0)$ of X defined by the nonsingular complex curve germ represented by φ at ζ_0 . By assumption, there exists some $1 \leq j_0 \leq N$ such that ω_{j_0} is nonzero at P_0 . It follows that the function $\text{eval}_{\text{id}_\mathbb{C}}(\varphi^* \omega_{j_0})$ associated to $\varphi^* \omega_{j_0}$ as described in (0.13) is nonzero at ζ_0 . Let η be a Kähler form of X and let $r_1 > r_0$. By (5.4) we have

$$T(r, r_1, \varphi, \eta) = O(\log r) \quad \parallel,$$

which implies that the holomorphic map φ from $\mathbb{C} - \overline{\Delta_{r_0}}$ to X can be extended to a holomorphic map from $\mathbb{C} \cup \{\infty\} - \overline{\Delta_{r_0}}$ to X . \square

Theorem 5.8. *For any integer $n \geq 3$ there exists a positive integer δ_n with the following property. For any positive integer $\delta \geq \delta_n$ there exists a proper subvariety \mathcal{Z} in the moduli space \mathbb{P}_N of all hypersurfaces of degree δ in \mathbb{P}_n (where $N = \binom{n+\delta}{n}$) such that for $\alpha \in \mathbb{P}_n - \mathcal{Z}$ and any holomorphic map $\varphi : \mathbb{C} - \overline{\Delta_{r_0}} \rightarrow X^{(\alpha)}$ (where $r_0 > 0$) can be extended to a holomorphic map $\mathbb{C} \cup \{\infty\} - \overline{\Delta_{r_0}} \rightarrow X^{(\alpha)}$, where $X^{(\alpha)}$ is the hypersurface of degree δ in \mathbb{P}_n corresponding to the point α in the moduli space \mathbb{P}_N .*

Proof. By Proposition (2.17) on global generation on jet space by slanted vector fields at points representable by regular curve germs, \square

II. ENTIRE FUNCTION SOLUTIONS OF POLYNOMIAL EQUATIONS OF SLOWLY VARYING COEFFICIENTS

5.9. Historical Background, Osculation Condition, and Log-Pole Jet Differential.

Before the introduction of the language of geometry of manifolds, hyperbolicity problems were formulated in terms of entire functions satisfying functional equations. For example, a theorem of Borel states that if entire function $\varphi_1, \dots, \varphi_n$ satisfy $e^{\varphi_1} + \dots + e^{\varphi_n} = 0$, then $\varphi_j - \varphi_k$ is constant with $1 \leq j < k \leq n$. In the formulation in terms of functional equations satisfied by entire functions, the hyperbolicity of a generic hypersurface of high degree δ states that no $n+1$ entire holomorphic functions $\varphi_0(\zeta), \dots, \varphi_n(\zeta)$, with the ratios $\frac{\varphi_j}{\varphi_\ell}$ not all constant, can satisfy a homogeneous polynomial equation

$$\sum_{\nu_0 + \dots + \nu_n = \delta} \alpha_{\nu_0, \dots, \nu_n} \varphi_0^{\nu_0} \dots \varphi_n^{\nu_n} \equiv 0$$

of degree δ whose constant coefficients $\alpha_{\nu_0, \dots, \nu_n}$ are generic.

There have also been considerable investigations on the situation when the constant coefficients are allowed to vary slowly. For example, on p.387 of his 1897 paper [Bo97], Emile Borel studied the problem of entire functions $\gamma_1(\zeta), \dots, \gamma_n(\zeta)$ and $\varphi_1(\zeta), \dots, \varphi_n(\zeta)$ satisfying $\gamma_1 e^{\varphi_1} + \dots + \gamma_n e^{\varphi_n} = 0$ and proved that $\gamma_1(\zeta), \dots, \gamma_n(\zeta)$ must be all identically zero if the growth rate on $|\zeta| = r$ of $\gamma_1(\zeta), \dots, \gamma_n(\zeta)$ and $\varphi_1(\zeta), \dots, \varphi_n(\zeta)$ is no more than $e^{\mu(|\zeta|)}$ while the growth rate on $|\zeta| = r$ of $\varphi_j(\zeta) - \varphi_\ell(\zeta)$ for $j \neq \ell$ is at least $\mu(r)^2$ for some function $\mu(r)$ as $r \rightarrow \infty$.

For the hyperbolicity problem of generic hypersurface of degree δ , we now study the question of entire functions satisfying a homogeneous polynomial equation of degree δ with varying coefficients. More precisely, we ask whether there are entire functions $\varphi_0(\zeta), \dots, \varphi_n(\zeta)$ without common zeroes and entire functions $\alpha_{\nu_0, \dots, \nu_n}(\zeta)$ for $\nu_0 + \dots + \nu_n = \delta$ without any common zeroes satisfying

$$(5.9.1)_0 \quad \sum_{\nu_0 + \dots + \nu_n = \delta} \alpha_{\nu_0, \dots, \nu_n}(\zeta) \varphi_0(\zeta)^{\nu_0} \dots \varphi_n(\zeta)^{\nu_n} \equiv 0$$

such that

- (i) $\psi : \zeta \mapsto \alpha(\zeta) = (\alpha_{\nu_0, \dots, \nu_n}(\zeta)) \in \mathbb{P}_N$ is nonconstant,
- (ii) $\alpha(\zeta_0) = (\alpha_{\nu_0, \dots, \nu_n}(\zeta_0))$ is not in the exceptional set \mathcal{Z} in \mathbb{P}_N for some $\zeta_0 \in \mathbb{C}$, and
- (iii) $T(r, \psi) = o(T(r, \varphi) + \log r)$ ||, where $\varphi : \mathbb{C} \rightarrow \mathbb{P}_n$ is defined by $[\varphi_0, \dots, \varphi_n]$.

Here we handle the simpler question which assumes in addition that

$$(5.9.1)_j \quad \sum_{\nu_0 + \dots + \nu_n = \delta} \alpha_{\nu_0, \dots, \nu_n}(\zeta) \frac{d^j}{d\zeta^j} (\varphi_0(\zeta)^{\nu_0} \dots \varphi_n(\zeta)^{\nu_n}) \equiv 0$$

for $1 \leq j \leq n-1$. The additional set of $n-1$ equations $(5.9.1)_j$ for $1 \leq j \leq n-1$ is equivalent to the set of $n-1$ equations

$$(5.9.2)_j \quad \sum_{\nu_0 + \dots + \nu_n = \delta} \left(\frac{d^j}{d\zeta^j} \alpha_{\nu_0, \dots, \nu_n}(\zeta) \right) (\varphi_0(\zeta)^{\nu_0} \dots \varphi_n(\zeta)^{\nu_n}) \equiv 0$$

for $1 \leq j \leq n-1$, because of the equation $(5.9.1)_0$ itself and the result obtained by differentiating it j -times with respect to ζ .

A geometric interpretation of the conditions $(5.9.1)_0$ and $(5.9.2)_j$ for $1 \leq j \leq n-1$ is the following. When

$$\{\varphi_0(\zeta)^{\nu_0} \dots \varphi_n(\zeta)^{\nu_n}\}_{\nu_0 + \dots + \nu_n = \delta}$$

is considered as the set of coefficients of a linear equation which defines a hyperplane $H(\zeta)$ in \mathbb{P}_N , as ζ varies in \mathbb{C} we have a moving hyperplane depending on ζ . Having entire functions $\varphi_\ell(\zeta)$ for $0 \leq \ell \leq n$ and their functions $\alpha_{\nu_0, \dots, \nu_n}(\zeta)$ for $\nu_0 + \dots + \nu_n = \delta$ satisfying $(5.9.1)_j$ for $0 \leq j \leq n-1$ is equivalent to the existence of a holomorphic map

$$\zeta \mapsto \alpha(\zeta) = (\alpha_{\nu_0, \dots, \nu_n}(\zeta))_{\nu_0 + \dots + \nu_n = \delta}$$

from \mathbb{C} to \mathbb{P}_N which osculates the hyperplane $H(\zeta)$ to order $n-1$ in the sense that the curve $\zeta \mapsto \alpha(\zeta)$ in \mathbb{P}_N is tangential to order $n-1$ to the hyperplane $H(\zeta_0)$ of \mathbb{P}_N at the point $\alpha(\zeta_0) \in \mathbb{P}_N$. Condition (iii) of $T(r, \psi)$ being of order $o(T(r, \varphi) + \log r)$ is the condition of slowly varying coefficients.

For this question of polynomial equations with slowly varying coefficients under additional assumption of osculation, we present here two results. The first one, corresponding to hyperbolicity, is that when the map $\zeta \mapsto \alpha(\zeta)$ is slowly moving compared to the map $\zeta \mapsto \varphi(\zeta) \in \mathbb{P}_n$, no such pair of curves $\zeta \mapsto \alpha(\zeta) \in \mathbb{P}_N$ and $\zeta \mapsto \varphi(\zeta) \in \mathbb{P}_n$ exist.

The second result, corresponding to the Big Picard Theorem, concerns extension across ∞ when the pair of maps $\zeta \mapsto \varphi(\zeta) \in \mathbb{P}_n$ and $\zeta \mapsto \alpha(\zeta) \in \mathbb{P}_N$ are only defined for $\zeta \in \mathbb{C} - \overline{\Delta_{r_0}}$ instead of on \mathbb{C} .

Since the Schwarz lemma is the crucial tool for the hyperbolicity problem, for the more general case of slowly varying coefficients we need a variation of the Schwarz lemma for it. We are going to present it in the form which is more than we need by allowing log-pole jet differentials rather than just holomorphic jet differentials so that it can be used later in this article in the proof of Second Main Theorems for log-pole jet differentials (see Theorem 5.15 and Theorem 5.16 below).

A log-pole jet differential means that locally it is of the form

$$\sum_{\lambda} G_{\lambda} (d^{\ell_{1,\lambda}} x_1)^{\nu_{1,\lambda}} \cdots (d^{\ell_{1,\lambda}} x_1)^{\nu_{1,\lambda}} (d^{\sigma_{1,\lambda}} \log F_1)^{\tau_{1,\lambda}} \cdots (d^{\sigma_{\mu_{\lambda},\lambda}} \log F_{\mu_{\lambda}})^{\tau_{\mu_{\lambda},\lambda}},$$

where x_1, \dots, x_n are local holomorphic coordinates, G_{λ} and $F_1, \dots, F_{\mu_{\lambda}}$ are local holomorphic functions. Each $(d^{\ell} \log F)^{\nu}$ contributes $\nu \ell$ times the divisor of F to the log-pole divisor (with multiplicities counted) of the log-pole jet differential.

Proposition 5.10. (General Schwarz Lemma for Log-Pole Differential on Subvariety of Jets and Map with Slow Growth for Pole Set) *Let X be a compact complex algebraic manifold of complex dimension n and Y be a complex subvariety of the space $J_k(X)$ of k -jets. Let $\pi_k : J_k(X) \rightarrow X$ be the natural projection map. Let D and E be nonnegative divisors of X whose associated line bundles L_D and L_E respectively have smooth metrics $e^{-\chi_D}$ and $e^{-\chi_E}$ with smooth $(1,1)$ -forms η_D and η_E as curvature forms such that $D + E$ is an ample divisor of X and its curvature form $\eta_D + \eta_E$ for the metric $e^{-\chi_D - \chi_E}$ is strictly positive on X . Let s_D (respectively s_E) be the holomorphic section of L_D (respectively L_E) whose divisor is D (respectively E). Let F be a nonnegative divisor of X and $\text{Supp } F$ be its support. Let ω be a function on Y such that $s_E (s_D)^{-1} \omega$ is holomorphic on $Y - \pi_k^{-1}(\text{Supp } F)$. Assume that for some finite open cover $\{U_j\}_{j=1}^J$ of X , there exists a log-pole k -form ω_j on U_j (for $1 \leq j \leq J$), whose log pole is contained in F with multiplicities counted, such that on $Y \cap \pi_k^{-1}(U_j)$ the function ω agrees with the function on $\pi_k^{-1}(U_j)$ defined by $s_D (s_E)^{-1} \omega_j$ for $1 \leq j \leq J$. Let $r_1 > r_0 > 0$. Let $\varphi : \mathbb{C} - \overline{\Delta_{r_0}} \rightarrow X$ be a holomorphic map such that the image of the map $J_k(\varphi) : J_k(\mathbb{C} - \overline{\Delta_{r_0}}) \rightarrow J_k(X)$ induced by φ is contained in Y . Let $G_j(\zeta)$ be the function $\text{eval}_{\text{id}_{\mathbb{C}}}(\varphi_j^* \omega)$ associated to $\varphi_j^* \omega$ as explained in (0.13). If $G_j(\zeta)$ is not identically zero on $\mathbb{C} - \overline{\Delta_{r_0}}$, then*

$$\begin{aligned} T(r, r_1, \varphi, \eta_D) &\leq T(r, r_1, \varphi, \eta_E) + N(r, r_1, \varphi, F) \\ &\quad + O(\log T(r, r_1, \varphi, \eta_D + \eta_E) + \log r) \quad \parallel. \end{aligned}$$

In particular, if for some $\varepsilon > 0$ one assumes that

$$N(r, r_1, \varphi, F) + T(r, r_1, \varphi, |\eta_E|) \leq (1 - \varepsilon) (T(r, r_1, \varphi, \eta_D)) \quad \parallel,$$

then either $G_j(\zeta)$ is identically zero for all $1 \leq j \leq J$ or

$$T(r, \varphi, \eta_D + \eta_E) = O(\log r) \quad \parallel.$$

Proof. We assume that $G_{j_0}(\zeta)$ is not identically zero for some $1 \leq j_0 \leq J$. We apply twice integration of Laplacian with in (0.12) to

$$g(\zeta) = \log \left(|G_{j_0}(\zeta)|^2 \varphi^* \left(\frac{|s_E|^2 e^{-\chi_E}}{|s_D|^2 e^{-\chi_D}} \right) \right).$$

Since ω_j is holomorphic on $Y - \pi_k^{-1}(\text{Supp } F)$, it follows that

$$\begin{aligned} & T(r, r_1, \varphi, \eta_D) - T(r, r_1, \varphi, \eta_E) - N(r, r_1, \varphi, F) \\ & \leq \frac{1}{4\pi} \int_{\theta=0}^{2\pi} \log \left(|G_{j_0}(\zeta)|^2 \varphi^* \left(\frac{|s_E|^2 e^{-\chi_E}}{|s_D|^2 e^{-\chi_D}} \right) \right) (re^{i\theta}) d\theta + O(1). \end{aligned}$$

Here we have the inequality instead of an identity, because of possible contribution from the zero-set of $\frac{\omega_{j_0}}{s_D}$. At this point enters Bloch's technique of applying the logarithmic derivative lemma by using the logarithm of global meromorphic functions as local coordinates. As functions on the space $J_{n-1}^{\text{vert}}(\mathcal{X})$ of vertical $(n-1)$ -jets on \mathcal{X} (with the right-hand side being global functions and the left-hand side being only local functions due to the transition functions of the line bundles L_D and L_E),

$$|\omega_{j_0} s_E s_D| \leq C \sum_{\lambda=1}^{\Lambda} \prod_{\nu_{j,\ell}, j, \ell} \left| d^\ell \log F_{j,\ell}^{(\lambda)} \right|^{\nu_{j,\ell}}$$

for some $C > 0$ and a finite collection $\{F_{j,\ell}^{(\lambda)}\}$ of global meromorphic functions on X , where the product is taken over the indices $\nu_{j,\ell}, j, \ell$ with the ranges $1 \leq j \leq n, 1 \leq \ell \leq k$ and $\sum_{1 \leq j \leq n, 1 \leq \ell \leq k} \ell \nu_{j,\ell} = m$. By Nevanlinna's logarithmic derivative lemma (extended to \mathbb{C} outside a disk centered at the origin) given in (5.3),

$$\int_{\theta=0}^{2\pi} \log^+ \left| d^\ell \log F_{j,\ell}^{(\lambda)} \right| (re^{i\theta}) d\theta = O(\log T(r, r_1, \varphi, \eta_D + \eta_E) + \log r) \quad \parallel.$$

Hence

$$\begin{aligned} & \int_{\theta=0}^{2\pi} \log \left(|G_{j_0}(\zeta)|^2 \varphi^* \left(\frac{|s_E|^2 e^{-\chi_E}}{|s_D|^2 e^{-\chi_D}} \right) \right) (re^{i\theta}) d\theta \\ & = O(\log T(r, r_1, \varphi, \eta_D + \eta_E) + \log r) \quad \parallel \end{aligned}$$

and we get

$$\begin{aligned} T(r, r_1, \varphi, \eta_D) & \leq T(r, r_1, \varphi, \eta_E) + N(r, r_1, \varphi, F) \\ & \quad + O(\log T(r, r_1, \varphi, \eta_D + \eta_E) + \log r) \quad \parallel. \end{aligned}$$

If now for some $\varepsilon > 0$ one assumes that

$$N(r, r_1, \varphi, F) + T(r, r_1, \varphi, |\eta_E|) \leq (1 - \varepsilon)(T(r, r_1, \varphi, \eta_D)) \quad \parallel,$$

then one obtains right away $T(r, \varphi, \eta_D + \eta_E) = O(\log r) \quad \parallel$. \square

Theorem 5.11. (Entire Function Solution of Polynomial Equations with Slowing Varying Coefficients) *There exists a positive integer δ_n and for $\delta \geq \delta_n$ there exists a property subvariety \mathcal{Z} of \mathbb{P}_N (where $N = \binom{\delta+n}{n}$) with the following property. There cannot exist entire functions $\varphi_0(\zeta), \dots, \varphi_n(\zeta)$ without common zeroes and entire functions $\alpha_{\nu_0, \dots, \nu_n}(\zeta)$ for $\nu_0 + \dots + \nu_n = \delta$ without any common zeroes satisfying*

$$\sum_{\nu_0 + \dots + \nu_n = \delta} \alpha_{\nu_0, \dots, \nu_n}(\zeta) \frac{d^j}{d\zeta^j} (\varphi_0(\zeta)^{\nu_0} \cdots \varphi_n(\zeta)^{\nu_n}) \equiv 0 \quad \text{for } 0 \leq j \leq n-1$$

such that

- (i) the map $\psi : \zeta \mapsto \alpha(\zeta) = (\alpha_{\nu_0, \dots, \nu_n}(\zeta)) \in \mathbb{P}_N$ is nonconstant,
- (ii) $\alpha(\zeta_0) = (\alpha_{\nu_0, \dots, \nu_n}(\zeta_0))$ is not in the exceptional set \mathcal{Z} in \mathbb{P}_N for some $\zeta_0 \in \mathbb{C}$, and
- (iii) $T(r, \psi) = o(T(r, \varphi) + \log r) \quad \parallel$, where $\varphi : \mathbb{C} \rightarrow \mathbb{P}_n$ is defined by $[\varphi_0, \dots, \varphi_n]$.

Proof. We apply Proposition 5.10 with $k = n-1$ to the the space $J_{n-1}(\mathcal{X})$ of $(n-1)$ -jets of the universal hypersurface \mathcal{X} with subvariety Y equal to the space $J_{n-1}^{\text{vert}}(\mathcal{X})$ of vertical $(n-1)$ -jets of \mathcal{X} . The assumption

$$\sum_{\nu_0 + \dots + \nu_n = \delta} \alpha_{\nu_0, \dots, \nu_n}(\zeta) \frac{d^j}{d\zeta^j} (\varphi_0(\zeta)^{\nu_0} \cdots \varphi_n(\zeta)^{\nu_n}) \equiv 0 \quad \text{for } 0 \leq j \leq n-1$$

implies that for every $\zeta \in \mathbb{C}$ the element of $J_{n-1}(\mathcal{X})$ represented by the parametrized complex curve germ φ at ζ belongs to $Y = J_{n-1}^{\text{vert}}(\mathcal{X})$.

By Proposition 4.3 we have a proper subvariety \mathcal{Z} of \mathbb{P}_N and for $\alpha \in \mathbb{P}_N - \mathcal{Z}$ and $1 \leq j \leq J$ a holomorphic family of $(n-1)$ -jet differentials $\omega_j^{(\alpha)}$ on $X^{(\alpha)}$ vanishing on the infinity hyperplane of \mathbb{P}_n (extendible to a meromorphic family over all of \mathbb{P}_N) such that, at any point P_0 of $J_{n-1}(X^{(\alpha)})$ with $\alpha \in \mathbb{P}_N - \mathcal{Z}$ which is representable by a nonsingular complex curve germ, at least one $\omega_j^{(\alpha)}$ is nonzero at P_0 for some $1 \leq j \leq J$.

Since for each $1 \leq j \leq J$ the holomorphic family $\omega_j^{(\alpha)}$ for $\alpha \in \mathbb{P}_N - \mathcal{Z}$ can be extended to a meromorphic family for α varying in all of \mathbb{P}_N , we can find a divisor E_j in \mathbb{P}_N such that for all $1 \leq j \leq J$ the pole-set of $\omega_j^{(\alpha)}$ as a meromorphic vertical $(n-1)$ -jet differential on \mathcal{X} is contained in the intersection of \mathcal{X} and $\mathbb{P}_n \times E_j$ with multiplicities counted. For $1 \leq j \leq J$, because $\omega_j^{(\alpha)}$ vanishes on an ample divisor of $X^{(\alpha)}$ for $\alpha \in \mathbb{P}_N - \mathcal{Z}$, we can find a divisor D_j in \mathcal{X} such that $D_j + E_j$ is an

ample divisor of \mathcal{X} and the zero-set of $\omega_j^{(\alpha)}$ as a meromorphic vertical $(n-1)$ -jet differential on \mathcal{X} contains D_j with multiplicities counted.

Since $\psi : \mathbb{C} \rightarrow \mathbb{P}_N$ is nonconstant and $T(r, \psi) = o(T(r, \varphi) + \log r) \parallel$, it follows that the differential $d\varphi$ is nonzero for some $\zeta_0 \in \mathbb{C}$. Denote by P_0 the point in $J_{n-1}(X^{(\alpha)})$ represented by the nonsingular complex curve germ φ at ζ_0 . Some $\omega_{j_0}^{(\alpha)}$ has nonzero value at P_0 . From Proposition 5.10 applied to $\omega_{j_0}^{(\alpha)}$, it follows that

$$T(r, \varphi) = O(\log r) \parallel,$$

which would contradict

$$\limsup_{r \rightarrow \infty} \frac{T(r, \psi)}{\log r} > 0$$

from the nonconstancy of the map ψ and the assumption

$$T(r, \psi) = o(T(r, \varphi) + \log r) \parallel.$$

□

The analogue of the Big Picard Theorem about removable essential singularities is the following result.

Theorem 5.12. (Removing Essential Singularity for Holomorphic Solution of Polynomial Equations with Slowly Varying Coefficients) *There exists a positive integer δ_n and for $\delta \geq \delta_n$ there exists a property subvariety \mathcal{Z} of \mathbb{P}_N (where $N = \binom{\delta+n}{n}$) with the following property. For some $r_0 > 0$ let $\varphi_0(\zeta), \dots, \varphi_n(\zeta)$ be holomorphic functions on $\mathbb{C} - \overline{\Delta_{r_0}}$ without common zeroes and let $\alpha_{\nu_0, \dots, \nu_n}(\zeta)$ for $\nu_0 + \dots + \nu_n = \delta$ be holomorphic functions on $\mathbb{C} - \overline{\Delta_{r_0}}$ without common zeroes. Assume that*

$$\sum_{\nu_0 + \dots + \nu_n = \delta} \alpha_{\nu_0, \dots, \nu_n}(\zeta) \frac{d^j}{d\zeta^j} (\varphi_0(\zeta)^{\nu_0} \dots \varphi_n(\zeta)^{\nu_n}) \equiv 0 \quad \text{for } 0 \leq j \leq n-1$$

on $\mathbb{C} - \overline{\Delta_{r_0}}$ such that

- (i) the map $\psi : \zeta \mapsto \alpha(\zeta) = (\alpha_{\nu_0, \dots, \nu_n}(\zeta)) \in \mathbb{P}_N$ is nonconstant,
- (ii) $\alpha(\zeta_0) = (\alpha_{\nu_0, \dots, \nu_n}(\zeta_0))$ is not in the exceptional set \mathcal{Z} in \mathbb{P}_N for some $\zeta_0 \in \mathbb{C} - \overline{\Delta_{r_0}}$, and
- (iii) $T(r, r_1, \psi) = o(T(r, r_1, \varphi) + \log r) \parallel$.

Then $\varphi_0(\zeta), \dots, \varphi_n(\zeta)$ and $\alpha_{\nu_0, \dots, \nu_n}(\zeta)$ for $\nu_0 + \dots + \nu_n = \delta$ can be extended to meromorphic functions on $\mathbb{C} - \overline{\Delta_{r_0}}$.

Proof. We use the same notations as in the proof of Theorem 5.11 except the domains for the maps $\varphi : \mathbb{C} - \overline{\Delta_{r_0}} \rightarrow \mathcal{X}$ and $\psi : \mathbb{C} - \overline{\Delta_{r_0}} \rightarrow \mathbb{P}_N$ are now different. Without loss of generality we can assume that the map φ is nonconstant, otherwise the extendibility of φ and ψ is clear. Denote by P_0 the point in $J_{n-1}(X^{(\alpha)})$ represented by the nonsingular complex curve germ φ at ζ_0 . Some $\omega_{j_0}^{(\alpha)}$ has nonzero value at P_0 . From Proposition 5.10 applied to $\omega_{j_0}^{(\alpha)}$, it follows that

$$T(r, \varphi) = O(\log r) \quad \parallel.$$

Now the extendibility of φ and ψ to respectively holomorphic maps $\mathbb{C} - \overline{\Delta_{r_0}} \rightarrow \mathcal{X}$ and $\mathbb{C} - \overline{\Delta_{r_0}} \rightarrow \mathbb{P}_N$ follows from Lemma 5.6. \square

III. SECOND MAIN THEOREM FROM LOG-POLE JET DIFFERENTIAL

Nevanlinna's Second Main Theorem is a quantitative version of the Little Picard Theorem. The hyperbolicity of generic hypersurface of high degree corresponds to the Little Picard Theorem. We now discuss the analogue of Nevanlinna's Second Main Theorem for any regular hypersurface of high degree from our approach of jet differentials.

In contrast to the use of holomorphic jet differentials vanishing on an ample divisor in the hyperbolicity problem, the jet differentials used for the Second Main Theorem are log-pole jet differentials vanishing on ample divisor. Our method fits in with Cartan's proof of the Second Main Theorem for entire holomorphic curves in \mathbb{P}_n and a collection of hyperplanes in \mathbb{P}_n in general position given in [Ca33].

We will first show how to construct log-pole jet differentials on \mathbb{P}_n which vanishes on an appropriate ample divisor of \mathbb{P}_n and whose log pole-set is contained in the hypersurface. We then present two Second Main Theorems for log-pole jet differentials, with the second one dealing with the situation of slowly moving targets. Then we show how Cartan's proof can be recast in our setting of Second Main Theorem for log-pole jet differentials vanishing on an appropriate ample divisor.

Second main theorems are useful only when the estimates are reasonably sharp. In the case at hand, because of our construction of jet differentials is so far away from the conjectured optimal situation, the discussion about Second Main Theorems can only serve as pointing out a connection between Second Main Theorems and jet differentials and their construction.

Theorem 5.13. (Existence of Log Pole Jet Differential) *Let $0 < \varepsilon_0, \varepsilon'_0 < 1$. There exists a positive integer $\hat{\delta}_n$ such that for any regular hypersurface X of degree $\delta \geq \hat{\delta}_n$ in \mathbb{P}_n there exists a non identically zero*

log-pole n -jet differential ω on \mathbb{P}_n of weight $\leq \delta^{\varepsilon_0}$ which vanishes with multiplicity at least $\delta^{1-\varepsilon'_0}$ on the infinity hyperplane of \mathbb{P}_n and which is holomorphic on $\mathbb{P}_n - X$. In particular, the log-pole divisor of ω is no more than λ times X with $\lambda \leq n\delta^{\varepsilon_0}$.

Proof. We choose $\epsilon, \epsilon', \theta_0, \theta, \theta'$ in the open interval $(0, 1)$ such that $(n+1)\theta_0 + \theta \geq (n+1) + \epsilon$, $1 - \epsilon'_0 \leq \theta' < 1 - \epsilon'$, and $\varepsilon_0 \leq \theta$. We apply Proposition 3.8 to get $A = A(n+1, \epsilon, \epsilon')$ from it and then set $\hat{\delta}_{n+1} = A$.

Let $f(x_1, \dots, x_n)$ be a polynomial in terms of the inhomogeneous coordinates x_1, \dots, x_n of \mathbb{P}_n which defines X . Let \hat{X} be the regular hypersurface in \mathbb{P}_{n+1} defined by the polynomial $F = f(x_1, \dots, x_n) - x_{n+1}^\delta$ in the inhomogeneous coordinates x_1, \dots, x_{n+1} of \mathbb{P}_{n+1} . We apply Proposition 3.8 to F (instead of to f) to get an n -jet differential $\hat{\omega}$ of the form $\frac{Q}{F_{x_1}-1}$ which vanishes to order $\geq \delta^{\theta'}$ at the infinity hyperplane of \mathbb{P}_{n+1} , where Q is a polynomial in

$$d^j x_1, \dots, d^j x_{n+1} \quad (0 \leq j \leq n)$$

which is of degree $m_0 = \lceil \delta^{\theta_0} \rceil$ in x_1, \dots, x_{n+1} and is of homogeneous weight $m = \lceil \delta^\theta \rceil$ in

$$d^j x_1, \dots, d^j x_{n+1} \quad (1 \leq j \leq n)$$

when the weight of $d^j x_\ell$ is assigned to be j .

We choose a nonzero integer ℓ such that one nonzero term of $\frac{Q}{x_{n+1}^\ell}$ is of the form

$$Q_0 \left(\frac{dx_{n+1}}{x_{n+1}} \right)^{\nu_1} \cdots \left(\frac{d^n x_{n+1}}{x_{n+1}} \right)^{\nu_n},$$

where Q_0 is a polynomial in the variables

$$d^j x_1, \dots, d^j x_n \quad (0 \leq j \leq n)$$

with constant coefficients and ν_1, \dots, ν_n are nonnegative integers.

The complex manifold \hat{X} is a branched cover over \mathbb{P}_n with cyclic branching of order δ at X under the projection map $\hat{\pi} : \hat{X} \rightarrow \mathbb{P}_n$ induced by $(x_1, \dots, x_n, x_{n+1}) \mapsto (x_1, \dots, x_n)$. Let ω be the direct image of

$$\frac{\hat{\omega}}{x_{n+1}^\ell} = \frac{Q}{x_{n+1}^\ell (f_{x_1} - 1)}$$

under $\hat{\pi}$. The n -jet differential ω on \mathbb{P}_n can be computed as follows. First we express $d^\ell x_{n+1}$ (by induction on ℓ) as a polynomial of the variables

$$x_{n+1}, d \log x_{n+1}, d^2 \log x_{n+1}, \dots, d^\ell \log x_{n+1}$$

with constant coefficients so that $\frac{Q}{x_{n+1}^\ell}$ is expressed as a polynomial of x_1, \dots, x_{n+1} and

$$d^j x_1, \dots, d^j x_n, d^j \log x_{n+1} \quad (0 \leq j \leq n)$$

with constant coefficients. Then we obtain ω from $\frac{Q}{x_{n+1}^\ell(f_{x_1}-1)}$ by replacing $d^j \log x_{n+1}$ by $d^j \log f$ and setting x_{n+1} equal to 0. The log-pole jet differential ω on \mathbb{P}_n is not identically zero, because of the nonzero term

$$Q_0 \left(\frac{dx_{n+1}}{x_{n+1}} \right)^{\nu_1} \cdots \left(\frac{d^n x_{n+1}}{x_{n+1}} \right)^{\nu_n},$$

in Q . The log-pole divisor of ω is no more than λ times X with $\lambda \leq n\delta^{\varepsilon_0}$, because ω is an n -jet differential of weight $\leq \delta^{\varepsilon_0}$. \square

Remark 5.14. Vanishing Order of Direct Image of Jet Differential. The log-pole jet differential constructed in Theorem 5.13 as the direct image of a holomorphic jet differential under the a branched cover map of δ sheets may in general have high vanishing order (comparable to δ) at points of \mathbb{P}_n , even though the polynomial Q on \hat{X} in

$$d^j x_1, \dots, d^j x_{n+1} \quad (0 \leq j \leq n)$$

used in the construction is of degree $m_0 = \lceil \delta^{\theta_0} \rceil$ in x_1, \dots, x_{n+1} and is of homogeneous weight $m = \lceil \delta^\theta \rceil$ in

$$d^j x_1, \dots, d^j x_{n+1} \quad (1 \leq j \leq n)$$

when the weight of $d^j x_\ell$ is assigned to be j . The reason is that a term like

$$Q_0 \left(\frac{dx_{n+1}}{x_{n+1}} \right)^{\nu_1} \cdots \left(\frac{d^n x_{n+1}}{x_{n+1}} \right)^{\nu_n},$$

would give rise to the term

$$Q_0 \left(\frac{df}{f} \right)^{\nu_1} \cdots \left(\frac{d^n f}{f} \right)^{\nu_n}$$

in its direct image and the derivatives of f may contribute to high vanishing order (comparable to δ).

However, when X is a generic hypersurface of \mathbb{P}_n , the arguments in the proof of Lemma 4.1 show that the vanishing order of the log-pole jet differential constructed in Theorem 5.13 as the direct image of a holomorphic jet differential has vanishing order comparable to $m_0 = \lceil \delta^{\theta_0} \rceil$ at points of $\mathbb{P}_n - X$.

By applying Proposition 5.10, we have the following two Second Main Theorems for log-pole jet differentials, with second one dealing with the case of slowly moving targets.

Theorem 5.15. (Second Main Theorem from Log-Pole Jet Differentials) *Let X be an n -dimensional compact complex manifold with an ample line bundle L . Let $D_1, \dots, D_p, E_1, \dots, E_q$ be divisors of L . Let ω be log-pole jet differential on X with vanishing on $D = D_1 + \dots + D_p$ such that the log-pole set of ω is contained in $E = E_1 + \dots + E_q$ with multiplicities counted. Then for any holomorphic map φ from the affine complex line \mathbb{C} to X such that the image of φ is not contained in E and the pullback $\varphi^*\omega$ not identically zero,*

$$pT(r, \varphi, L) \leq N(r, \varphi, E) + O(\log T(r, \varphi, L)) \quad \parallel.$$

In other words,

$$\sum_{j=1}^q m(r, \varphi, E_j) \leq (q-p)pT(r, \varphi, L) + O(\log T(r, \varphi, L)) \quad \parallel.$$

The meaning of the log-pole set of ω being contained in $E = E_1 + \dots + E_q$ with multiplicities counted is the following. Locally ω is of the form

$$\sum_{\tau_1, \lambda_1, \dots, \tau_k, \lambda_k} h_{\tau_1, \lambda_1, \dots, \tau_k, \lambda_k} (d^{\tau_1} \log F_1)^{\lambda_1} \dots (d^{\tau_\ell} \log F_\ell)^{\lambda_\ell}$$

with

$$\tau_1 \lambda_1 \operatorname{div} F_1 + \dots + \tau_\ell \lambda_\ell \operatorname{div} F_\ell$$

contained in E with multiplicities counted, where $\operatorname{div} F_j$ is the divisor of F_j .

Theorem 5.16. (Second Main Theorem for Jet Differential with Slowly Moving Targets) *Let $S \subset \mathbb{P}_N$ be a complex algebraic manifold and $\mathcal{X} \subset \mathbb{P}_{\hat{n}} \times S$ be a complex algebraic manifold. Let $\pi : \mathcal{X} \rightarrow S$ be the projection induced by the natural projection $\mathbb{P}_{\hat{n}} \times \mathbb{P}_N \rightarrow \mathbb{P}_N$ to the second factor. Let L_S be an ample line bundle of S . Let L be a line bundle of \mathcal{X} such that $L + \pi^{-1}(L_S)$ is ample on X . Let $D_1, \dots, D_p, E_1, \dots, E_q$ be divisors of L . Let $D = D_1 + \dots + D_p$ and $E = E_1 + \dots + E_q$. For $a \in S$ let $X^{(\alpha)} = \pi^{-1}(a)$ and $D^{(\alpha)} = D|_{X^{(\alpha)}}$ and $E^{(\alpha)} = E|_{X^{(\alpha)}}$. Let Z be a proper subvariety of S . For $a \in S - Z$ let $\omega^{(\alpha)}$ be a log-pole jet differential on $X^{(\alpha)}$ such that $\omega^{(\alpha)}$ vanishes on the divisor $D^{(\alpha)}$ and the log-pole set of $\omega^{(\alpha)}$ is contained in divisor $E^{(\alpha)}$ with multiplicity counted. Assume that $\omega^{(\alpha)}$ is holomorphic in a for $a \in S - Z$ and is meromorphic in a for $a \in S$. Let φ be a holomorphic map from the*

affine complex line \mathbb{C} to \mathcal{X} such that the image of $\pi \circ \varphi$ is not contained in Z and $T(r, \pi \circ \varphi, L_S) = o(T(r, \varphi, L + \pi^{-1}(L_S)))$, then

$$qT(r, \varphi, L + \pi^{-1}(L_S)) \leq N(r, \varphi, D) + o(T(r, \varphi, L + \pi^{-1}(L_S))) \quad \parallel.$$

In other words,

$$\sum_{j=1}^p m(r, \varphi, D_j) \leq (q-p)T(r, \varphi, L + \pi^{-1}(L_S)) + o(T(r, \varphi, L + \pi^{-1}(L_S))) \quad \parallel.$$

In the product case of $\mathcal{X} = X^{(0)} \times S$ with $D_j = D_j^{(0)}$, the proximity function $m(r, \varphi, D_j)$ in the formulation can be replaced $m(r, \text{proj}_1 \circ \varphi, D_j^{(0)})$, because $N(r, \varphi, D)$ is equal to $N(r, \text{proj}_1 \circ \varphi, D_j^{(0)})$ and we can apply Nevanlinna's First Main Theorem to $\text{proj}_1 \circ \varphi$ and the divisor $D_j^{(0)}$ and use the assumption $T(r, \pi \circ \varphi, L_S) = o(T(r, \varphi, L + \pi^{-1}(L_S)))$.

In the following remark we discuss how Cartan's proof of his Second Main Theorem for hyperplanes in general position can be interpreted in the setting of the Second Main Theorem for log-pole jet differentials.

Remark 5.17. Cartan's Second Main Theorem for hyperplanes in \mathbb{P}_n for hyperplanes in general position given in [Ca33] is simply the special case of Theorem 5.15 with

$$\omega = \frac{\text{Wron}(dx_1, \dots, dx_n)}{F_1 \cdots F_q}$$

in inhomogeneous coordinates x_1, \dots, x_n of \mathbb{P}_n , where F_1, \dots, F_q are the degree-one polynomial in x_1, \dots, x_n which define the q hyperplanes in \mathbb{P}_n in general position.

Here the notation for the Wronskian

$$\text{Wron}(\eta_1, \dots, \eta_\ell)$$

for jet differentials η_1, \dots, η_ℓ on a complex manifold Y is used to mean the jet differential

$$\det(d^{\lambda-1}\eta_j)_{1 \leq \lambda, j \leq \ell} = \sum_{\sigma \in S_\ell} (\text{sgn } \sigma) \eta_{\sigma(1)}(d\eta_{\sigma(2)}) \cdots (d^{\ell-1}\eta_{\sigma(\ell)})$$

on Y , where S_ℓ is the group of all permutations of $\{1, 2, \dots, \ell\}$ and $\text{sgn } \sigma$ is the signature of the permutation σ .

The denominator $F_1 \cdots F_q$ in ω gives the vanishing order q at the infinity hyperplane of \mathbb{P}_n . The key argument here is that from the general position assumption of the zero-sets of F_1, \dots, F_q we can locally write ω as a constant times

$$\frac{\text{Wron}(dF_{\nu_1}, \dots, dF_{\nu_n})}{F_1 \cdots F_q} = \frac{\text{Wron}(d \log F_{\nu_1}, \dots, d \log F_{\nu_n})}{F_{\nu_{n+1}} \cdots F_{\nu_q}}$$

in a neighborhood U of a point when F_j is nowhere zero on U for j not equal to any of the indices ν_1, \dots, ν_n .

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